

# HAMILTONIAN MECHANICS ON DUALS OF GENERALIZED LIE ALGEBROIDS

by  
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## Abstract

A new description, different by the classical theory of Hamiltonian Mechanics, in the general framework of generalized Lie algebroids is presented. In the particular case of Lie algebroids, new and important results are obtained. We present the *dual mechanical systems* called by use, *dual mechanical  $(\rho, \eta)$ -systems*, *Hamilton mechanical  $(\rho, \eta)$ -systems* or *Cartan mechanical  $(\rho, \eta)$ -systems*. We obtain the canonical  $(\rho, \eta)$ -semi(spray) associated to a dual mechanical  $(\rho, \eta)$ -system. The Hamilton mechanical  $(\rho, \eta)$ -systems are the spaces necessary to develop a Hamiltonian formalism. We obtain the  $(\rho, \eta)$ -semispray associated to a regular Hamiltonian  $H$  and external force  $F_e$  and we derive the equations of Hamilton-Jacobi type.

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**Keywords:** fiber bundle, vector bundle, (generalized) Lie algebroid, (linear) connection, curve, lift, natural base, adapted base, projector, almost product structure, almost tangent structure, complex structure, spray, semispray, dual mechanical system, Hamiltonian formalism.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Preliminaries</b>	<b>3</b>
<b>3</b>	<b>Natural and adapted basis</b>	<b>6</b>
<b>4</b>	<b>The lift of a differentiable curve</b>	<b>11</b>
<b>5</b>	<b>Remarkable Mod-endomorphisms</b>	<b>13</b>
5.1	Projectors . . . . .	14
5.2	The almost product structure . . . . .	16
5.3	The almost tangent structure . . . . .	17
<b>6</b>	<b>Tensor <math>d</math>-fields. Distinguished linear <math>(\rho, \eta)</math>-connections</b>	<b>17</b>
<b>7</b>	<b>Dual mechanical systems</b>	<b>22</b>
<b>8</b>	<b><math>(\rho, \eta)</math>-semisprays and <math>(\rho, \eta)</math>-sprays for dual mechanical <math>(\rho, \eta)</math>-systems</b>	<b>24</b>
<b>9</b>	<b>A Hamiltonian formalism for Hamilton mechanical <math>(\rho, \eta)</math>-systems</b>	<b>32</b>
	<b>References</b>	<b>35</b>

# 1 Introduction

The concept of Hamilton space, introduced in [15], was intensively studied in [6, 7, 8, 11, 14], and it has been successful, as a geometric theory of the Hamiltonian function. In the classical sense, a regular Hamiltonian on  $T^*M$  is a smooth function  $T^*M \xrightarrow{H} \mathbb{R}$  such that the Hessian matrix with entries

$$g^{ij}(x, p) = \frac{1}{2} \frac{\partial^2 H(x, p)}{\partial p_i \partial p_j}$$

is everywhere nondegenerate on  $T^*M$  (or on a domain of  $T^*M$ ) and a Hamilton space is a pair  $H^n = (M, H)$ , where  $H$  is a regular Hamiltonian. (see [16]) The case when  $H$  is square of a function on  $T^*M$ , positively, 1-homogeneous with respect to the momentum  $p_i$ , provides an important class of Hamilton spaces called Cartan spaces. The modern formulation of the geometry of Cartan spaces was given by R. Miron [12, 13] although some results were obtained by É. Cartan [5] and A. Kawaguchi [9].

The geometry of  $T^*M$  is from one point of view different from that of  $TM$ , because not exists a natural tangent structure and a semispray can not be introduced as usual for the tangent bundle. Two geometrical ingredients are of great importance on  $T^*M$ : the canonical 1-form  $p_i dx^i$  and its exterior derivative  $dp_i \wedge dx^i$  (the canonical symplectic structure of  $T^*M$ ). They are systematically used to defined new useful tools in the classical theory.

A Hamiltonian description of Mechanics on duals of Lie algebroids was presented in [10]. (see also [17, 18, 19, 20, 21]) The role of cotangent bundle of the configuration manifold was played by the prolongation  $\mathcal{L}^{\tau^*}E$  of  $E$  along the projection  $E^* \xrightarrow{\tau^*} M$ . The Lie algebroid version of the classical results concerning the universality of the standard Liouville 1-form on cotangent bundles is presented in *Theorem 3.4* and *Corollary 3.6*. Given a Hamiltonian function  $E^* \xrightarrow{H} \mathbb{R}$  and the symplectic form  $\Omega_E$  on  $E^*$ , the dynamics are obtained solving the equation

$$i_{\xi_H} \Omega_E = d\mathcal{L}^{\tau^*} E H$$

with the usual notations. The solutions of  $\xi_H$  (curves in  $E^*$ ) are the ones of the Hamilton equations for  $H$ .

The purpose of the present paper is to find the answer to the following question:

- *Could we to give a Hamiltonian description of Mechanics on duals of generalized Lie algebroids (see [1, 2, 3]) similar with the Lagrangian description of Mechanics on generalized Lie algebroids presented in the paper [4] without the symplectic form?*

In Sections 3, 4, 5 and 6 we set up the basic notions and terminology. In Section 7 we present for the first time the *dual mechanical systems* called by use, *dual mechanical*  $(\rho, \eta)$ -systems, *Hamilton mechanical*  $(\rho, \eta)$ -systems or *Cartan mechanical*  $(\rho, \eta)$ -systems.

In Section 8 we obtain the *canonical*  $(\rho, \eta)$ -semispray associated to the dual mechanical  $(\rho, \eta)$ -system  $\left( \left( E, \pi, M \right), F_e, (\rho, \eta) \Gamma \right)$  and from locally invertible  $\mathbf{B}^{\mathbf{v}}$ -morphism  $(g, h)$ . Also, we present the *canonical*  $(\rho, \eta)$ -spray associated to mechanical system  $\left( \left( E, \pi, M \right), F_e, (\rho, \eta) \Gamma \right)$  and from locally invertible  $\mathbf{B}^{\mathbf{v}}$ -morphism  $(g, h)$ .

The Section 9 is dedicated to study the geometry of Hamilton mechanical  $(\rho, \eta)$ -systems. These mechanical systems are the spaces necessary to obtain a Hamiltonian formalism in the general framewok of generalized Lie algebroids. We determine and we study the  $(\rho, \eta)$ -semispray associated to a regular Hamiltonian  $H$  and external force  $\bar{F}_e^*$  which are applied on the dual of the total space of a generalized Lie algebroid and we derive the equations of Hamilton-Jacobi type.

Finally, we obtain that the integral curves of the canonical  $(\rho, \eta)$ -semispray associated to Hamilton mechanical  $(\rho, \eta)$ -system  $\left( \left( \bar{E}^*, \bar{\pi}^*, M \right), \bar{F}_e^*, (\rho, \eta) \Gamma \right)$  and from locally invertible  $\mathbf{B}^\mathbf{v}$ -morphism  $(g, h)$  are the  $(g, h)$ -lifts solutions for the equations of Hamilton-Jacobi type (9.10).

Our researches are very important because, if  $h = Id_M = \eta$ , then all results presented in this paper become new results in the framework of Lie algebroids.

## 2 Preliminaries

Let **Vect**, **Liealg**, **Mod**, **Man** and  $\mathbf{B}^\mathbf{v}$  be the category of real vector spaces, Lie algebras, modules, manifolds and vector bundles respectively.

We know that if  $(E, \pi, M) \in |\mathbf{B}^\mathbf{v}|$  so that  $M$  is paracompact and if  $A \subseteq M$  is closed, then for any section  $u$  over  $A$  it exists  $\tilde{u} \in \Gamma(E, \pi, M)$  so that  $\tilde{u}|_A = u$ . In the following, we consider only vector bundles with paracompact base.

Additionally, if  $(E, \pi, M) \in |\mathbf{B}^\mathbf{v}|$ ,  $\Gamma(E, \pi, M) = \{u \in \mathbf{Man}(M, E) : u \circ \pi = Id_M\}$  and  $\mathcal{F}(M) = \mathbf{Man}(M, \mathbb{R})$ , then  $(\Gamma(E, \pi, M), +, \cdot)$  is a  $\mathcal{F}(M)$ -module. If  $(\varphi, \varphi_0) \in \mathbf{B}^\mathbf{v}((E, \pi, M), (E', \pi', M'))$  such that  $\varphi_0 \in Iso_{\mathbf{Man}}(M, M')$ , then, using the operation

$$\begin{array}{ccc} \mathcal{F}(M) \times \Gamma(E', \pi', M') & \xrightarrow{\quad} & \Gamma(E', \pi', M') \\ (f, u') & \mapsto & f \circ \varphi_0^{-1} \cdot u' \end{array}$$

it results that  $(\Gamma(E', \pi', M'), +, \cdot)$  is a  $\mathcal{F}(M)$ -module and we obtain the **Mod**-morphism

$$\begin{array}{ccc} \Gamma(E, \pi, M) & \xrightarrow{\Gamma(\varphi, \varphi_0)} & \Gamma(E', \pi', M') \\ u & \mapsto & \Gamma(\varphi, \varphi_0) u \end{array}$$

defined by

$$\Gamma(\varphi, \varphi_0) u(y) = \varphi \left( u_{\varphi_0^{-1}(y)} \right),$$

for any  $y \in M'$ .

Let  $M, N \in |\mathbf{Man}|$ ,  $h \in Iso_{\mathbf{Man}}(M, N)$  and  $\eta \in Iso_{\mathbf{Man}}(N, M)$ .

We know (see [2, 3]) that if  $(F, \nu, N) \in |\mathbf{B}^\mathbf{v}|$  so that there exists

$$(\rho, \eta) \in \mathbf{B}^\mathbf{v}((F, \nu, N), (TM, \tau_M, M))$$

and an operation

$$\begin{array}{ccc} \Gamma(F, \nu, N) \times \Gamma(F, \nu, N) & \xrightarrow{[\cdot]_{F, h}} & \Gamma(F, \nu, N) \\ (u, v) & \mapsto & [u, v]_{F, h} \end{array}$$

with the following properties:

$GLA_1$ . the equality holds good

$$[u, f \cdot v]_{F,h} = f[u, v]_{F,h} + \Gamma(Th \circ \rho, h \circ \eta)(u) f \cdot v,$$

for all  $u, v \in \Gamma(F, \nu, N)$  and  $f \in \mathcal{F}(N)$ .

$GLA_2$ . the 4-tuple  $(\Gamma(F, \nu, N), +, \cdot, [, ]_{F,h})$  is a Lie  $\mathcal{F}(N)$ -algebra,

$GLA_3$ . the **Mod**-morphism  $\Gamma(Th \circ \rho, h \circ \eta)$  is a **LieAlg**-morphism of

$$(\Gamma(F, \nu, N), +, \cdot, [, ]_{F,h})$$

source and

$$(\Gamma(TN, \tau_N, N), +, \cdot, [, ]_{TN})$$

target, then the triple  $((F, \nu, N), [, ]_{F,h}, (\rho, \eta))$  is called generalized Lie algebroid.

In particular, if  $h = Id_M = \eta$ , then we obtain the definition of the Lie algebroid.

We can discuss about *the category **GLA** of generalized Lie algebroids*. (see [3])

Examples of objects of this category are presented in the paper [2].

Let  $((F, \nu, N), [, ]_{F,h}, (\rho, \eta))$  be an object of the category **GLA**.

- Locally, for any  $\alpha, \beta \in \overline{1, p}$ , we set  $[t_\alpha, t_\beta]_{F,h} = L_{\alpha\beta}^\gamma t_\gamma$ . We easily obtain that  $L_{\alpha\beta}^\gamma = -L_{\beta\alpha}^\gamma$ , for any  $\alpha, \beta, \gamma \in \overline{1, p}$ .

The real local functions  $L_{\alpha\beta}^\gamma$ ,  $\alpha, \beta, \gamma \in \overline{1, p}$  will be called the *structure functions of the generalized Lie algebroid*  $((F, \nu, N), [, ]_{F,h}, (\rho, \eta))$ .

- We assume the following diagrams:

$$\begin{array}{ccccc} F & \xrightarrow{\rho} & TM & \xrightarrow{Th} & TN \\ \downarrow \nu & & \downarrow \tau_M & & \downarrow \tau_N \\ N & \xrightarrow{\eta} & M & \xrightarrow{h} & N \\ (\chi^{\tilde{i}}, z^\alpha) & & (x^i, y^i) & & (\chi^{\tilde{i}}, z^{\tilde{i}}) \end{array}$$

where  $i, \tilde{i} \in \overline{1, m}$  and  $\alpha \in \overline{1, p}$ .

If

$$\begin{aligned} (\chi^{\tilde{i}}, z^\alpha) &\longrightarrow (\chi^{\tilde{i}'}(\chi^{\tilde{i}}), z^{\alpha'}(\chi^{\tilde{i}}, z^\alpha)), \\ (x^i, y^i) &\longrightarrow (x^{\tilde{i}'}(x^i), y^{\tilde{i}'}(x^i, y^i)) \end{aligned}$$

and

$$(\chi^{\tilde{i}}, z^{\tilde{i}}) \longrightarrow (\chi^{\tilde{i}'}(\chi^{\tilde{i}}), z^{\tilde{i}'}(\chi^{\tilde{i}}, z^{\tilde{i}})),$$

then

$$z^{\alpha'} = \Lambda_\alpha^{\alpha'} z^\alpha,$$

$$y^{\tilde{i}'} = \frac{\partial x^{\tilde{i}'}}{\partial x^i} y^i$$

and

$$z^{\tilde{i}'} = \frac{\partial \chi^{\tilde{i}'}}{\partial \chi^{\tilde{i}}} z^{\tilde{i}}.$$

- We assume that  $(\theta, \mu) \stackrel{put}{=} (Th \circ \rho, h \circ \eta)$ . If  $z^\alpha t_\alpha \in \Gamma(F, \nu, N)$  is arbitrary, then

$$(2.1) \quad \begin{aligned} & \Gamma(Th \circ \rho, h \circ \eta)(z^\alpha t_\alpha) f(h \circ \eta(\varkappa)) = \\ & = \left( \theta_\alpha^i z^\alpha \frac{\partial f}{\partial \varkappa^i} \right) (h \circ \eta(\varkappa)) = \left( (\rho_\alpha^i \circ h)(z^\alpha \circ h) \frac{\partial f \circ h}{\partial x^i} \right) (\eta(\varkappa)), \end{aligned}$$

for any  $f \in \mathcal{F}(N)$  and  $\varkappa \in N$ .

The coefficients  $\rho_\alpha^i$  respectively  $\theta_\alpha^{\tilde{i}}$  change to  $\rho_{\alpha'}^{\check{i}}$  respectively  $\theta_{\alpha'}^{\check{i}}$  according to the rule:

$$(2.2) \quad \rho_{\alpha'}^{\check{i}} = \Lambda_\alpha^\alpha \rho_\alpha^i \frac{\partial x^{\check{i}}}{\partial x^i},$$

respectively

$$(2.3) \quad \theta_{\alpha'}^{\check{i}} = \Lambda_\alpha^\alpha \theta_\alpha^{\tilde{i}} \frac{\partial \varkappa^{\check{i}}}{\partial \varkappa^{\tilde{i}}},$$

where

$$\|\Lambda_{\alpha'}^\alpha\| = \|\Lambda_\alpha^\alpha\|^{-1}.$$

*Remark 2.1* The following equalities hold good:

$$(2.4) \quad \rho_\alpha^i \circ h \frac{\partial f \circ h}{\partial x^i} = \left( \theta_\alpha^{\tilde{i}} \frac{\partial f}{\partial \varkappa^{\tilde{i}}} \right) \circ h, \forall f \in \mathcal{F}(N).$$

and

$$(2.5) \quad \left( L_{\alpha\beta}^\gamma \circ h \right) \left( \rho_\gamma^k \circ h \right) = (\rho_\alpha^i \circ h) \frac{\partial (\rho_\beta^k \circ h)}{\partial x^i} - (\rho_\beta^j \circ h) \frac{\partial (\rho_\alpha^k \circ h)}{\partial x^j}.$$

Let  $(E, \pi, M) \in |\mathbf{B}^v|$  and  $\left( \overset{*}{E}, \overset{*}{\pi}, \overset{*}{M} \right)$  its dual. We have the  $\mathbf{B}^v$ -morphism

$$(2.6) \quad \begin{array}{ccc} \overset{*}{\pi}^* (h^* F) & \hookrightarrow & F \\ \overset{*}{\pi}^* (h^* \nu) \downarrow & & \downarrow \nu \\ M & \xrightarrow{h \circ \overset{*}{\pi}} & N \end{array}$$

Let  $\left( \overset{*}{\pi}^* (h^* F), \overset{*}{\pi}^* (h^* \nu), \overset{*}{E} \right)$  be the  $\mathbf{B}^v$ -morphism of  $\left( \overset{*}{\pi}^* (h^* F), \overset{*}{\pi}^* (h^* \nu), \overset{*}{E} \right)$  source and  $\left( \overset{*}{T}E, \overset{*}{\tau}_E, \overset{*}{E} \right)$  target, where

$$(2.7) \quad \begin{array}{ccc} \overset{*}{\pi}^* (h^* F) & \xrightarrow{\overset{*}{\rho}} & \overset{*}{T}E \\ Z^\alpha T_\alpha \left( \overset{*}{u}_x \right) & \longmapsto & \left( Z^\alpha \cdot \rho_\alpha^i \circ h \circ \overset{*}{\pi} \right) \frac{\partial}{\partial x^i} \left( \overset{*}{u}_x \right) \end{array}$$

Using the operation

$$\Gamma \left( \overset{*}{\pi}^* (h^* F), \overset{*}{\pi}^* (h^* \nu), \overset{*}{E} \right)^2 \xrightarrow{[\cdot]_{\overset{*}{\pi}^* (h^* F)}} \Gamma \left( \overset{*}{\pi}^* (h^* F), \overset{*}{\pi}^* (h^* \nu), \overset{*}{E} \right)$$

defined by

$$\begin{aligned}
(2.8) \quad [T_\alpha, T_\beta]_{\pi^{**}(h^*F)} &= \left( L_{\alpha\beta}^\gamma \circ h \circ \pi^* \right) T_\gamma, \\
[T_\alpha, fT_\beta]_{\pi^{**}(h^*F)} &= f \left( L_{\alpha\beta}^\gamma \circ h \circ \pi^* \right) T_\gamma + \left( \rho_\alpha^i \circ h \circ \pi^* \right) \frac{\partial f}{\partial x^i} T_\beta, \\
[fT_\alpha, T_\beta]_{\pi^{**}(h^*F)} &= -[T_\beta, fT_\alpha]_{\pi^{**}(h^*F)},
\end{aligned}$$

for any  $f \in \mathcal{F} \left( \overset{*}{E} \right)$ , it results that

$$\left( \left( \pi^{**}(h^*F), \pi^{**}(h^*\nu), \overset{*}{E} \right), [\cdot, \cdot]_{\pi^{**}(h^*F)}, \left( \pi^{**}(h^*F), Id_{\overset{*}{E}} \right) \right)$$

is a Lie algebroid.

### 3 Natural and adapted basis

In the following we consider the following diagram:

$$(3.1) \quad \begin{array}{ccc} \overset{*}{E} & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \pi^* \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where  $(E, \pi, M) \in |\mathbf{B}^\vee|$  and  $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta))$  is a generalized Lie algebroid.

Let  $(\rho, \eta) \Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $(\overset{*}{E}, \pi^*, M)$ .

We take  $(x^i, p_a)$  as canonical local coordinates on  $(\overset{*}{E}, \pi^*, M)$ , where  $i \in \overline{1, m}$  and  $a \in \overline{1, r}$ . Let

$$(x^i, p_a) \longrightarrow (x^{\check{i}}(x^i), p_{a'}(x^i, p_a))$$

be a change of coordinates on  $(\overset{*}{E}, \pi^*, M)$ . Then the coordinates  $p_a$  change to  $p_{a'}$  by the rule:

$$(3.2) \quad p_{a'} = M_{a'}^a p_a.$$

Let

$$(3.3) \quad \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial p_a} \right) \overset{put}{=} \left( \overset{*}{\partial}_i, \overset{\cdot a}{\partial} \right)$$

be the natural base of the dual tangent Lie algebroid  $\left( \left( TE, \tau_E^*, \overset{*}{E} \right), [\cdot, \cdot]_{TE^*}, \left( Id_{TE^*}, Id_{\overset{*}{E}} \right) \right)$ .

For any sections

$$Z^\alpha T_\alpha \in \Gamma \left( \pi^{**}(h^*F), \pi^{**}(h^*F), \overset{*}{E} \right)$$

and

$$Y_a \overset{\cdot a}{\partial} \in \Gamma \left( VT E^*, \tau_E^*, E^* \right)$$

we obtain the section

$$\begin{aligned} Z^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a \overset{\cdot a}{\tilde{\partial}} &=: Z^\alpha \left( T_\alpha \oplus \left( \rho_\alpha^i \circ h \circ \pi^* \right) \overset{*}{\partial}_i \right) + Y_a \left( 0_{\pi^*(h^*F)}^* \oplus \overset{\cdot a}{\partial} \right) \\ &= Z^\alpha T_\alpha \oplus \left( Z^\alpha \left( \rho_\alpha^i \circ h \circ \pi^* \right) \overset{*}{\partial}_i + Y_a \overset{\cdot a}{\partial} \right) \in \Gamma \left( \pi^*(h^*F) \oplus T E^*, \pi^*, E^* \right). \end{aligned}$$

Since we have

$$\begin{aligned} Z^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a \overset{\cdot a}{\tilde{\partial}} &= 0 \\ \Downarrow \\ Z^\alpha T_\alpha = 0 \wedge Z^\alpha \left( \rho_\alpha^i \circ h \circ \pi^* \right) \overset{*}{\partial}_i + Y_a \overset{\cdot a}{\partial} &= 0, \end{aligned}$$

it implies  $Z^\alpha = 0$ ,  $\alpha \in \overline{1, p}$  and  $Y_a = 0$ ,  $a \in \overline{1, r}$ .

Therefore, the sections  $\overset{*}{\tilde{\partial}}_1, \dots, \overset{*}{\tilde{\partial}}_p, \overset{\cdot 1}{\tilde{\partial}}, \dots, \overset{\cdot r}{\tilde{\partial}}$  are linearly independent.

We consider the vector subbundle  $\left( (\rho, \eta) T E^*, (\rho, \eta) \tau_E^*, E^* \right)$  of the vector bundle  $\left( \pi^*(h^*F) \oplus T E^*, \pi^*, E^* \right)$ , for which the  $\mathcal{F} \left( E^* \right)$ -module of sections is the  $\mathcal{F} \left( E^* \right)$ -submodule of  $\left( \Gamma \left( \pi^*(h^*F) \oplus T E^*, \pi^*, E^* \right), +, \cdot \right)$ , generated by the set of sections  $\left( \overset{*}{\tilde{\partial}}_\alpha, \overset{\cdot a}{\tilde{\partial}} \right)$  which is called the *natural*  $(\rho, \eta)$ -base.

The matrix of coordinate transformation on  $\left( (\rho, \eta) T E^*, (\rho, \eta) \tau_E^*, E^* \right)$  at a change of fibred charts is

$$(3.4) \quad \left\| \begin{array}{cc} \Lambda_{\alpha'}^{\alpha} \circ h \circ \pi^* & 0 \\ \left( \rho_a^i \circ h \circ \pi^* \right) \frac{\partial M_b^{\alpha'} \circ \pi^*}{\partial x_i} y^b & M_a^{\alpha'} \circ \pi^* \end{array} \right\|.$$

We have the following

**Theorem 3.1** Let  $\left( \overset{*}{\tilde{\rho}}, Id_E^* \right)$  be the  $\mathbf{B}^v$ -morphism of  $\left( (\rho, \eta) T E^*, (\rho, \eta) \tau_E^*, E^* \right)$  source and  $\left( T E^*, \tau_E^*, E^* \right)$  target, where

$$(3.5) \quad \begin{aligned} &(\rho, \eta) T E^* \xrightarrow{\overset{*}{\tilde{\rho}}} T E^* \\ &\left( Z^\alpha \overset{*}{\tilde{\partial}}_\alpha + Y_a \overset{\cdot a}{\tilde{\partial}} \right) (\overset{*}{u}_x) \longmapsto \left( Z^\alpha \left( \rho_\alpha^i \circ h \circ \pi^* \right) \overset{*}{\partial}_i + Y_a \overset{\cdot a}{\partial} \right) (\overset{*}{u}_x) \end{aligned}$$

Using the operation

$$\Gamma \left( (\rho, \eta) T E^*, (\rho, \eta) \tau_E^*, E^* \right)^2 \xrightarrow{[\cdot]_{(\rho, \eta) T E^*}} \Gamma \left( (\rho, \eta) T E^*, (\rho, \eta) \tau_E^*, E^* \right)$$

defined by

$$\begin{aligned}
(3.6) \quad & \left[ \left( Z_1^\alpha \tilde{\partial}_\alpha^* + Y_a^1 \tilde{\partial}^{\cdot a} \right), \left( Z_2^\beta \tilde{\partial}_\beta^* + Y_b^2 \tilde{\partial}^{\cdot b} \right) \right]_{(\rho, \eta) TE^*} \\
&= \left[ Z_1^\alpha T_a, Z_2^\beta T_\beta \right]_{\pi^{**}(h^* F)} \oplus \left[ \left( \rho_\alpha^i \circ h \circ \pi^* \right) Z_1^\alpha \partial_i^* + Y_a^1 \tilde{\partial}^{\cdot a}, \right. \\
&\quad \left. \left( \rho_\beta^j \circ h \circ \pi^* \right) Z_2^\beta \partial_j^* + Y_b^2 \tilde{\partial}^{\cdot b} \right]_{TE^*},
\end{aligned}$$

for any  $\left( Z_1^\alpha \tilde{\partial}_\alpha^* + Y_a^1 \tilde{\partial}^{\cdot a} \right)$  and  $\left( Z_2^\beta \tilde{\partial}_\beta^* + Y_b^2 \tilde{\partial}^{\cdot b} \right)$ , we obtain that the couple

$$\left( [\cdot, \cdot]_{(\rho, \eta) TE^*}, \left( \tilde{\rho}^*, Id_E^* \right) \right)$$

is a Lie algebroid structure for the vector bundle  $\left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$ .

The Lie algebroid

$$\left( \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right), [\cdot, \cdot]_{(\rho, \eta) TE^*}, \left( \tilde{\rho}^*, Id_E^* \right) \right),$$

is called the *Lie algebroid generalized tangent bundle of dual vector bundle*  $\left( E^*, \pi^*, M \right)$ .

*Remark 3.1* The following equalities hold good:

$$\begin{aligned}
(3.7) \quad & \left[ \tilde{\partial}_\alpha^*, \tilde{\partial}_\beta^* \right]_{(\rho, \eta) TE^*} = \left( L_{\alpha\beta}^\gamma \circ h \circ \pi^* \right) \tilde{\partial}_\gamma^* \\
& \left[ \tilde{\partial}_\alpha^*, \tilde{\partial}^{\cdot b} \right]_{(\rho, \eta) TE^*} = 0_{(\rho, \eta) TE^*} \\
& \left[ \tilde{\partial}^{\cdot a}, \tilde{\partial}^{\cdot b} \right]_{(\rho, \eta) TE^*} = 0_{(\rho, \eta) TE^*}
\end{aligned}$$

We consider the  $\mathbf{B}^v$ -morphism  $\left( (\rho, \eta) \pi^!, Id_E^* \right)$  given by the commutative diagram

$$\begin{array}{ccc}
(\rho, \eta) TE^* & \xrightarrow{(\rho, \eta) \pi^!} & \pi^{**}(h^* F) \\
(\rho, \eta) \tau_E^* \downarrow & & \downarrow pr_1 \\
E^* & \xrightarrow{id_E^*} & E^*
\end{array}$$

Using the components, this is defined as:

$$(3.9) \quad (\rho, \eta) \pi^! \left( \tilde{Z}^\alpha \tilde{\partial}_\alpha^* + Y_a^1 \tilde{\partial}^{\cdot a} \right) (u_x) = \left( \tilde{Z}^\alpha \tilde{T}_\alpha \right) (u_x),$$

for any  $\tilde{Z}^\alpha \tilde{\partial}_\alpha^* + Y_a^1 \tilde{\partial}^{\cdot a} \in \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$ .



Using the  $\mathbf{B}^v$ -morphism  $\left((\rho, \eta)^* \pi^!, Id_E^*\right)$  and the  $\mathbf{B}^v$ -morphism (2.7) we obtain the *tangent*  $(\rho, \eta)$ -application  $\left((\rho, \eta)^* T\pi^*, h \circ \pi^*\right)$  of  $\left((\rho, \eta)^* TE, (\rho, \eta)^* \tau_E^*, E\right)$  source and  $(F, \nu, N)$  target.

Using the  $\mathbf{B}^v$ -morphisms (2.6) and (3.7) we obtain the *tangent*  $(\rho, \eta)$ -application  $\left((\rho, \eta)^* T\pi^*, h \circ \pi^*\right)$  of  $\left((\rho, \eta)^* TE, (\rho, \eta)^* \tau_E^*, E\right)$  source and  $(F, \nu, N)$  target.

**Definition 3.1** The kernel of the tangent  $(\rho, \eta)$ -application is written

$$\left(V(\rho, \eta)^* TE, (\rho, \eta)^* \tau_E^*, E\right)$$

and is called *the vertical subbundle*.

We remark that the set  $\left\{\dot{\tilde{\partial}}^a, a \in \overline{1, r}\right\}$  is a base of the  $\mathcal{F}\left(\overset{*}{E}\right)$ -module

$$\left(\Gamma\left(V(\rho, \eta)^* TE, (\rho, \eta)^* \tau_E^*, E\right), +, \cdot\right).$$

**Proposition 3.1** *The short sequence of vector bundles*

$$(3.10) \quad \begin{array}{ccccccccc} 0 & \xrightarrow{i} & V(\rho, \eta)^* TE & \xrightarrow{i} & (\rho, \eta)^* TE & \xrightarrow{(\rho, \eta)^* \pi^!} & \pi^* (h^* F) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \overset{*}{E} & \xrightarrow{Id_E^*} & \overset{*}{E} & \xrightarrow{Id_E^*} & \overset{*}{E} & \xrightarrow{Id_E^*} & \overset{*}{E} & \xrightarrow{Id_E^*} & \overset{*}{E} \end{array}$$

is exact.

Let  $(\rho, \eta) \Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $\left(\overset{*}{E}, \pi^*, M\right)$ , i. e. a **Man**-morphism of  $(\rho, \eta)^* TE$  source and  $V(\rho, \eta)^* TE$  target defined by

$$(3.11) \quad (\rho, \eta) \Gamma \left( \tilde{Z}^\alpha \dot{\tilde{\partial}}_\alpha^* + Y_b \dot{\tilde{\partial}}^b \right) \left( \overset{*}{u}_x \right) = \left( Y_b - (\rho, \eta) \Gamma_{b\alpha} \tilde{Z}^\alpha \right) \dot{\tilde{\partial}}^b \left( \overset{*}{u}_x \right),$$

such that the  $\mathbf{B}^v$ -morphism  $\left((\rho, \eta) \Gamma, Id_E^*\right)$  is a split to the left in the previous exact sequence. Its components satisfy the law of transformation

$$(3.12) \quad (\rho, \eta) \Gamma_{b\gamma'} = M_b^b \circ \pi^* \left[ - \left( \rho_\gamma^i \circ h \circ \pi^* \right) \frac{\partial M_b^{\alpha' \circ \pi^*}}{\partial x^i} p_{\alpha'} + (\rho, \eta) \Gamma_{b\gamma} \right] \left( \Lambda_{\gamma'}^\gamma \circ h \circ \pi^* \right).$$

The kernel of the  $\mathbf{B}^v$ -morphism  $\left((\rho, \eta) \Gamma, Id_E^*\right)$  is written  $\left(H(\rho, \eta)^* TE, (\rho, \eta)^* \tau_E^*, E\right)$  and is called *the horizontal vector subbundle*.

We remark that the horizontal and the vertical vector subbundles are interior differential systems of the Lie algebroid generalized tangent bundle

$$\left( \left( (\rho, \eta)^* TE, (\rho, \eta)^* \tau_E^*, E \right), [\cdot, \cdot]_{(\rho, \eta)^* TE}, \left( \overset{*}{\rho}, Id_E^* \right) \right).$$

We put the problem of finding a base for the  $\mathcal{F}\left(\overset{*}{E}\right)$ -module

$$\left(\Gamma\left(H(\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right), +, \cdot\right)$$

of the type

$$\overset{*}{\tilde{\delta}}_{\alpha} = Z_{\alpha}^{\beta} \overset{*}{\tilde{\delta}}_{\beta} + Y_{a\alpha} \overset{\cdot}{\tilde{\partial}}^a, \alpha \in \overline{1, r}$$

which satisfies the following conditions:

$$\begin{aligned} \Gamma\left((\rho, \eta) \overset{*}{\pi}!, Id_{\overset{*}{E}}\right) \left(\overset{*}{\tilde{\delta}}_{\alpha}\right) &= T_{\alpha}, \\ \Gamma\left((\rho, \eta) \Gamma, Id_{\overset{*}{E}}\right) \left(\overset{*}{\tilde{\delta}}_{\alpha}\right) &= 0. \end{aligned} \quad (3.13)$$

Then we obtain the sections

$$\overset{*}{\tilde{\delta}}_{\alpha} = \overset{*}{\tilde{\partial}}_{\alpha} + (\rho, \eta) \Gamma_{b\alpha} \overset{\cdot}{\tilde{\partial}}^b = T_{\alpha} \oplus \left( \left( \rho_{\alpha}^i \circ h \circ \overset{*}{\pi} \right) \overset{*}{\partial}_i - (\rho, \eta) \Gamma_{b\alpha} \overset{\cdot}{\partial}^b \right). \quad (3.14)$$

such that their law of change is a tensorial law under a change of vector fiber charts.

The base  $\left(\overset{*}{\tilde{\delta}}_{\alpha}, \overset{\cdot}{\tilde{\partial}}^a\right)$  will be called the *adapted*  $(\rho, \eta)$ -base.

*Remark 3.2* The following equality holds good

$$\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right) \left(\overset{*}{\tilde{\delta}}_{\alpha}\right) = \left(\rho_{\alpha}^i \circ h \circ \overset{*}{\pi}\right) \overset{*}{\partial}_i - (\rho, \eta) \Gamma_{b\alpha} \overset{\cdot}{\partial}^b. \quad (3.15)$$

Moreover, if  $(\rho, \eta) \Gamma$  is the  $(\rho, \eta)$ -connection associated to a connection  $\Gamma$  (see [1]), then we obtain

$$\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right) \left(\overset{*}{\tilde{\delta}}_{\alpha}\right) = \left(\rho_{\alpha}^i \circ h \circ \overset{*}{\pi}\right) \overset{*}{\delta}_i, \quad (3.16)$$

where  $\left(\overset{*}{\delta}_i, \overset{\cdot}{\partial}^a\right)$  is the adapted base for the  $\mathcal{F}\left(\overset{*}{E}\right)$ -module  $\left(\Gamma\left(T\overset{*}{E}, \tau_{\overset{*}{E}}, \overset{*}{E}\right), +, \cdot\right)$ .

**Theorem 3.2** *The following equality holds good*

$$\left[\overset{*}{\tilde{\delta}}_{\alpha}, \overset{*}{\tilde{\delta}}_{\beta}\right]_{(\rho, \eta) T\overset{*}{E}} = \left(L_{\alpha\beta}^{\gamma} \circ h \circ \overset{*}{\pi}\right) \overset{*}{\tilde{\delta}}_{\gamma} + (\rho, \eta, h) \mathbb{R}_{b\alpha\beta} \overset{\cdot}{\tilde{\partial}}^b, \quad (3.17)$$

where

$$\begin{aligned} (\rho, \eta, h) \mathbb{R}_{b\alpha\beta} &= \Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right) \left(\overset{*}{\tilde{\delta}}_{\beta}\right) ((\rho, \eta) \Gamma_{b\alpha}) \\ &+ \Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right) \left(\overset{*}{\tilde{\delta}}_{\alpha}\right) ((\rho, \eta) \Gamma_{b\beta}) - \left(L_{\alpha\beta}^{\gamma} \circ h \circ \overset{*}{\pi}\right) (\rho, \eta) \Gamma_{b\gamma}, \end{aligned} \quad (3.18)$$

Moreover, we have:

$$\left[\overset{*}{\tilde{\delta}}_{\alpha}, \overset{\cdot}{\tilde{\partial}}^a\right]_{(\rho, \eta) T\overset{*}{E}} = -\Gamma\left(\overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}}\right) \left(\overset{\cdot}{\tilde{\partial}}^a\right) ((\rho, \eta) \Gamma_{b\alpha}) \overset{\cdot}{\tilde{\partial}}^b, \quad (3.19)$$

and

$$(3.20) \quad \Gamma \left( \begin{smallmatrix} * \\ \tilde{\rho}, Id_E^* \end{smallmatrix} \right) \left[ \begin{smallmatrix} * \\ \tilde{\delta}_\alpha, \tilde{\delta}_\beta \end{smallmatrix} \right]_{(\rho, \eta) T E^*} = \left[ \Gamma \left( \begin{smallmatrix} * \\ \tilde{\rho}, Id_E^* \end{smallmatrix} \right) \left( \begin{smallmatrix} * \\ \tilde{\delta}_\alpha \end{smallmatrix} \right), \Gamma \left( \begin{smallmatrix} * \\ \tilde{\rho}, Id_E^* \end{smallmatrix} \right) \left( \begin{smallmatrix} * \\ \tilde{\delta}_\beta \end{smallmatrix} \right) \right]_{T E^*}.$$

Let  $(d\tilde{z}^\alpha, d\tilde{p}_a)$  be the natural dual  $(\rho, \eta)$ -base of natural  $(\rho, \eta)$ -base  $\left( \begin{smallmatrix} * \\ \partial_\alpha, \tilde{\partial}^a \end{smallmatrix} \right)$ .

This is determined by the equations

$$\begin{cases} \left\langle d\tilde{z}^\alpha, \begin{smallmatrix} * \\ \tilde{\partial}_\beta \end{smallmatrix} \right\rangle = \delta_\beta^\alpha, & \left\langle d\tilde{z}^\alpha, \begin{smallmatrix} \cdot \\ \tilde{\partial}^b \end{smallmatrix} \right\rangle = 0, \\ \left\langle d\tilde{p}_a, \begin{smallmatrix} * \\ \tilde{\partial}_\beta \end{smallmatrix} \right\rangle = 0, & \left\langle d\tilde{p}_a, \begin{smallmatrix} \cdot \\ \tilde{\partial}^b \end{smallmatrix} \right\rangle = \delta_a^b. \end{cases}$$

We consider the problem of finding a base for the  $\mathcal{F} \left( \begin{smallmatrix} * \\ E \end{smallmatrix} \right)$ -module

$$\left( \Gamma \left( \left( V(\rho, \eta) T E^* \right)^*, \left( (\rho, \eta) \tau_{E^*}^*, E \right), +, \cdot \right) \right)$$

of the type

$$\delta \tilde{p}_a = \theta_{a\alpha} d\tilde{z}^\alpha + \omega_a^b d\tilde{p}_b, \quad a \in \overline{1, r}$$

which satisfies the following conditions:

$$(3.21) \quad \left\langle \delta \tilde{p}_a, \begin{smallmatrix} \cdot \\ \tilde{\partial}^b \end{smallmatrix} \right\rangle = \delta_a^b \wedge \left\langle \delta \tilde{p}_a, \begin{smallmatrix} * \\ \tilde{\delta}_\alpha \end{smallmatrix} \right\rangle = 0,$$

We obtain the sections

$$(3.22) \quad \delta \tilde{p}_a = -(\rho, \eta) \Gamma_{a\alpha} d\tilde{z}^\alpha + d\tilde{p}_a, \quad a \in \overline{1, r}.$$

such that their changing rule is tensorial under a change of vector fiber charts. The base  $(d\tilde{z}^\alpha, \delta \tilde{p}_a)$  will be called the *adapted dual  $(\rho, \eta)$ -base*.

## 4 The lift of a differentiable curve

We consider the following diagram:

$$(4.1) \quad \begin{array}{ccc} \begin{smallmatrix} * \\ E \end{smallmatrix} & & (F, [\cdot, \cdot]_{F,h}, (\rho, \eta)) \\ \begin{smallmatrix} * \\ \pi \end{smallmatrix} \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where  $(E, \pi, M) \in |\mathbf{B}^\vee|$  and  $((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)) \in |\mathbf{GLA}|$ .

We admit that  $(\rho, \eta) \Gamma$  is a  $(\rho, \eta)$ -connection for the vector bundle  $\left( \begin{smallmatrix} * \\ E, \pi, M \end{smallmatrix} \right)$  and  $I \xrightarrow{c} M$  is a differentiable curve. We know that

$$\left( \begin{smallmatrix} * \\ E|_{\text{Im}(\eta \circ h \circ c)}, \pi|_{\text{Im}(\eta \circ h \circ c)}, \text{Im}(\eta \circ h \circ c) \end{smallmatrix} \right)$$

is a vector subbundle of the vector bundle  $\left(E, \pi, M\right)^*$ .

**Definition 4.1** If

$$(4.2) \quad \begin{array}{ccc} I & \xrightarrow{\dot{c}} & E|_{\text{Im}(\eta \circ h \circ c)}^* \\ t & \longmapsto & p_a(t) s^a(\eta \circ h \circ c(t)) \end{array}$$

is a differentiable curve such that there exists  $g \in \mathbf{Man}\left(E, F\right)^*$  such that the following conditions are satisfied:

1.  $(g, h) \in \mathbf{B}^v\left(\left(E, \pi, M\right)^*, (F, \nu, N)\right)$  and
2.  $\rho \circ g \circ \dot{c}(t) = \frac{d(\eta \circ h \circ c)^i(t)}{dt} \frac{\partial}{\partial x^i}((\eta \circ h \circ c)(t))$ , for any  $t \in I$ , then we will say that  $\dot{c}$  is the  $(g, h)$ -lift of the differentiable curve  $c$ .

*Remark 4.1* The condition 2 is equivalent with the following affirmation:

$$(4.3) \quad \rho_\alpha^i(\eta \circ h \circ c(t)) g^{\alpha a}(h \circ c(t)) p_a(t) = \frac{d(\eta \circ h \circ c)^i(t)}{dt}, \quad i \in \overline{1, m}.$$

**Definition 4.2** If  $I \xrightarrow{\dot{c}} E|_{\text{Im}(\eta \circ h \circ c)}^*$  is a differentiable  $(g, h)$ -lift of the differentiable curve  $c$ , then the section

$$(4.4) \quad \begin{array}{ccc} \text{Im}(\eta \circ h \circ c) & \xrightarrow{\dot{u}(c, \dot{c})} & E|_{\text{Im}(\eta \circ h \circ c)}^* \\ \eta \circ h \circ c(t) & \longmapsto & \dot{c}(t) \end{array}$$

will be called the *canonical section associated to the couple*  $(c, \dot{c})$ .

**Definition 4.3** If  $(g, h) \in \mathbf{B}^v\left(\left(E, \pi, M\right)^*, (F, \nu, N)\right)$  has the components

$$g^{\alpha a}; a \in \overline{1, r}, \quad \alpha \in \overline{1, p}$$

such that for any vector local  $(n+p)$ -chart  $(V, t_V)$  of  $(F, \nu, N)$  there exists the real functions

$$V \xrightarrow{\tilde{g}_{a\alpha}} \mathbb{R}; \quad a \in \overline{1, r}, \quad \alpha \in \overline{1, p}$$

such that

$$(4.4) \quad \tilde{g}_{a\alpha}(\varkappa) \cdot g^{\alpha b}(\varkappa) = \delta_a^b, \quad \forall \varkappa \in V,$$

then we will say that the  $\mathbf{B}^v$ -morphism  $(g, h)$  is locally invertible.

*Remark 4.2* In particular, if  $(Id_{TM}, Id_M, Id_M) = (\rho, \eta, h)$  and the  $\mathbf{B}^v$  morphism  $(g, Id_M)$  is locally invertible, then we have the differentiable  $(g, Id_M)$ -lift

$$(4.6) \quad \begin{array}{ccc} I & \xrightarrow{\dot{c}} & TM \\ t & \longmapsto & \tilde{g}_{ji}(c(t)) \frac{dc^j(t)}{dt} dx^i(c(t)) \end{array}.$$

**Definition 4.4** If  $I \xrightarrow{\dot{c}} \overset{*}{E}|_{\text{Im}(\eta \circ h \circ c)}$  is a differentiable  $(g, h)$ -lift for the curve  $c$  such that its components functions  $(p_b, b \in \overline{1, r})$  are solutions for the differentiable system of equations:

$$(4.7) \quad \frac{du_b}{dt} + (\rho, \eta) \Gamma_{b\alpha} \circ \overset{*}{u}(c, \dot{c}) \circ (\eta \circ h \circ c) \cdot g^{\alpha a} \circ h \circ c \cdot u_a = 0,$$

then we will say that the  $(g, h)$ -lift  $\dot{c}$  is parallel with respect to the  $(\rho, \eta)$ -connection  $(\rho, \eta) \overset{*}{\Gamma}$ .

*Remark 4.3*  $\left( \tilde{g}_{ji} \circ c \cdot \frac{dc^i}{dt}, j \in \overline{1, m} \right)$  are solutions for the differentiable system of equations

$$(4.8) \quad \frac{du_j}{dt} + \Gamma_{jk} \circ \overset{*}{u}(c, \dot{c}) \circ c \cdot g^{kh} \circ c \cdot u_h = 0,$$

namely

$$(4.8)' \quad \frac{d}{dt} \left( \tilde{g}_{ji} \circ c(t) \cdot \frac{dc^i(t)}{dt} \right) + \Gamma_{jk} \left( c(t), \left( \tilde{g}_{ji} \circ c(t) \cdot \frac{dc^i(t)}{dt} \right) \cdot dx^i(c(t)) \right) \cdot \frac{dc^k(t)}{dt} = 0,$$

## 5 Remarkable Mod-endomorphisms

Now, let us consider the following diagram:

$$\begin{array}{ccc} \overset{*}{E} & & (F, [, ]_{F,h}, (\rho, \eta)) \\ \overset{*}{\pi} \downarrow & & \downarrow \nu \\ M & \xrightarrow{h} & N \end{array}$$

where  $(E, \pi, M) \in |\mathbf{B}^v|$  and  $((F, \nu, N), [, ]_{F,h}, (\rho, \eta))$  is a generalized Lie algebroid.

Let  $(\rho, \eta) \Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $(\overset{*}{E}, \overset{*}{\pi}, M)$

**Definition 5.1** For any **Mod**-endomorphism  $e$  of

$$\left( \Gamma \left( (\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right), +, \cdot \right)$$

we define the application of Nijenhuis type

$$\Gamma \left( (\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)^2 \xrightarrow{N_e} \Gamma \left( (\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$$

defined by

$$N_e(X, Y) = [eX, eY]_{(\rho, \eta) T\overset{*}{E}} + e^2 [X, Y]_{(\rho, \eta) T\overset{*}{E}} - e[eX, Y]_{(\rho, \eta) T\overset{*}{E}} - e[X, eY]_{(\rho, \eta) T\overset{*}{E}},$$

for any  $X, Y \in \Gamma \left( (\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ .

## 5.1 Projectors

**Definition 5.1.1** Any **Mod**-endomorphism  $e$  of  $\Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$  with the property

$$(5.1.1) \quad e^2 = e$$

will be called *projector*.

**Example 5.1.1** The **Mod**-endomorphism

$$\begin{aligned} \Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right) &\xrightarrow{\mathcal{V}^*} \Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right) \\ \tilde{Z}^\alpha \tilde{\delta}_\alpha^* + Y_a \tilde{\partial}^{\cdot a} &\longmapsto Y_a \tilde{\partial}^{\cdot a} \end{aligned}$$

is a projector which will be called the *vertical projector*.

*Remark 5.1.1* We have  $\mathcal{V}^* \left( \tilde{\delta}_\alpha^* \right) = 0$  and  $\mathcal{V}^* \left( \tilde{\partial}^{\cdot a} \right) = \tilde{\partial}^{\cdot a}$ . Therefore, it follows

$$\mathcal{V}^* \left( \tilde{\partial}_\alpha^* \right) = -(\rho, \eta) \Gamma_{b\alpha} \tilde{\partial}^{\cdot b}.$$

In addition, we obtain the equality

$$(5.1.2) \quad \Gamma \left( (\rho, \eta) \Gamma, Id_E \right) \left( Z^\alpha \tilde{\partial}_\alpha^* + Y_a \tilde{\partial}^{\cdot a} \right) = \mathcal{V} \left( Z^\alpha \tilde{\partial}_\alpha^* + Y_a \tilde{\partial}^{\cdot a} \right),$$

for any  $Z^\alpha \tilde{\partial}_\alpha^* + Y_a \tilde{\partial}^{\cdot a} \in \Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right)$ .

**Theorem 5.1.1** A  $(\rho, \eta)$ -connection for the vector bundle  $\left( E^*, \pi^*, M \right)$  is characterized by the existence of a **Mod**-endomorphism  $\mathcal{V}^*$  of  $\left( \Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right), +, \cdot \right)$  with the properties:

$$(5.1.3) \quad \begin{aligned} \mathcal{V}^* \left( \Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right) \right) &\subset \Gamma \left( \left( V(\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right) \right) \\ \mathcal{V}^*(X) = X &\iff X \in \Gamma \left( \left( V(\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right) \right) \end{aligned}$$

**Example 5.1.2** The **Mod**-endomorphism

$$\begin{aligned} \Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right) &\xrightarrow{\mathcal{H}^*} \Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E^* \right) \\ \tilde{Z}^\alpha \tilde{\delta}_\alpha^* + Y_a \tilde{\partial}^{\cdot a} &\longmapsto \tilde{Z}^\alpha \tilde{\delta}_\alpha^* \end{aligned}$$

is a projector which will be called the *horizontal projector*.

*Remark 5.1.2* We have  $\mathcal{H}^* \left( \tilde{\delta}_\alpha^* \right) = \tilde{\delta}_\alpha^*$  and  $\mathcal{H}^* \left( \tilde{\partial}^{\cdot a} \right) = 0$ . Therefore, we obtain  $\mathcal{H}^* \left( \tilde{\partial}_\alpha^* \right) = \tilde{\delta}_\alpha^*$ .

**Theorem 5.1.2** A  $(\rho, \eta)$ -connection for the vector bundle  $\left(E, \pi, M\right)^*$  is characterized by the existence of a **Mod**-endomorphism  $\mathcal{H}^*$  of

$$\left(\Gamma\left((\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E\right), +, \cdot\right)$$

with the properties:

$$(5.1.4) \quad \begin{aligned} \Gamma\left((\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E\right) &\subset \Gamma\left(H(\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E\right) \\ \mathcal{H}^*(X) = X &\iff X \in \Gamma\left(H(\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E\right). \end{aligned}$$

**Corollary 5.1.1** A  $(\rho, \eta)$ -connection for the vector bundle  $\left(E, \pi, M\right)^*$  is characterized by the existence of a **Mod**-endomorphism  $\mathcal{H}^*$  of

$$\left(\Gamma\left((\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E\right), +, \cdot\right)$$

with the properties:

$$(5.1.5) \quad \begin{aligned} \mathcal{H}^{*2} &= \mathcal{H}^* \\ \text{Ker}\left(\mathcal{H}^*\right) &= \Gamma\left(V(\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E\right), +, \cdot. \end{aligned}$$

*Remark 5.1.3* For any

$$X \in \Gamma\left((\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E\right)$$

we obtain the following unique decomposition

$$X = \mathcal{H}^*X + \mathcal{V}^*X.$$

**Proposition 5.1.1** After some calculations we obtain

$$(5.1.6) \quad N_{\mathcal{V}}^*(X, Y) = \mathcal{V}^*\left[\mathcal{H}^*X, \mathcal{H}^*Y\right]_{(\rho, \eta)TE^*} = N_{\mathcal{H}^*}^*(X, Y),$$

for any  $X, Y \in \Gamma\left((\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E\right)$ .

**Corollary 5.1.2** The horizontal interior differential system

$$\left(H(\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E\right)$$

is involutive if and only if  $N_{\mathcal{V}}^* = 0$  or  $N_{\mathcal{H}^*}^* = 0$ .

## 5.2 The almost product structure

**Definition 5.2.1** Any **Mod**-endomorphism  $e$  of

$$\left( \Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E \right), +, \cdot \right)$$

with the property

$$(5.2.1) \quad e^2 = Id$$

will be called the *almost product structure*.

**Example 5.2.1** The **Mod**-endomorphism

$$\begin{aligned} \Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E \right) &\xrightarrow{\tilde{\mathcal{P}}} \Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E \right) \\ \tilde{Z}^\alpha \tilde{\delta}_\alpha^* + Y_a \tilde{\partial}^{\cdot a} &\longmapsto \tilde{Z}^\alpha \tilde{\delta}_\alpha^* - Y_a \tilde{\partial}^{\cdot a} \end{aligned}$$

is an almost product structure.

*Remark 5.2.1* The previous almost product structure has the properties:

$$(5.2.2) \quad \begin{aligned} \tilde{\mathcal{P}} &= 2\tilde{\mathcal{H}} - Id; \\ \tilde{\mathcal{P}} &= Id - 2\tilde{\mathcal{V}}; \\ \tilde{\mathcal{P}} &= \tilde{\mathcal{H}} - \tilde{\mathcal{V}}. \end{aligned}$$

*Remark 5.2.2* We obtain that  $\tilde{\mathcal{P}} \left( \tilde{\delta}_\alpha^* \right) = \tilde{\delta}_\alpha^*$  and  $\tilde{\mathcal{P}} \left( \tilde{\partial}^{\cdot a} \right) = -\tilde{\partial}^{\cdot a}$ . Therefore, it follows

$$\tilde{\mathcal{P}} \left( \tilde{\partial}_\alpha^* \right) = \tilde{\delta}_\alpha^* - \rho \Gamma_{b\alpha}^{\cdot b} \tilde{\partial}^{\cdot b}.$$

**Theorem 5.2.1** A  $(\rho, \eta)$ -connection for the vector bundle  $\left( E, \pi^*, M \right)$  is characterized by the existence of a **Mod**-endomorphism  $\tilde{\mathcal{P}}$  of

$$\left( \Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E \right), +, \cdot \right)$$

with the following property:

$$(5.2.3) \quad \tilde{\mathcal{P}}(X) = -X \iff X \in \Gamma \left( V(\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E \right).$$

**Proposition 5.2.1** After some calculations, we obtain

$$N_{\tilde{\mathcal{P}}}^*(X, Y) = 4\tilde{\mathcal{V}} \left[ \tilde{\mathcal{H}}X, \tilde{\mathcal{H}}Y \right],$$

for any  $X, Y \in \Gamma \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E \right)$ .

**Corollary 5.2.1** The horizontal interior differential system  $\left( H(\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E \right)$  is involutive if and only if  $N_{\tilde{\mathcal{P}}}^* = 0$ .



### 5.3 The almost tangent structure

**Definition 5.3.1** Any **Mod**-endomorphism  $e$  of  $\left(\Gamma((\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E^*)\right)$  with the property

$$(5.3.1) \quad e^2 = 0$$

will be called the *almost tangent structure*.

**Example 5.3.1** If  $(E, \pi, M) = (F, \nu, N)$ ,  $g \in \mathbf{Man}\left(E, E\right)^*$  such that  $(g, h)$  is a locally invertible **B<sup>v</sup>**-morphism, then the **Mod**-endomorphism

$$\begin{aligned} \Gamma\left((\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E^*\right) &\xrightarrow{\mathcal{J}_{(g, h)}^*} \Gamma\left((\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E^*\right) \\ \tilde{Z}^a \tilde{\partial}_a^* + Y_b \tilde{\partial}^{\cdot b} &\longmapsto \left(\tilde{g}_{ba} \circ h \circ \pi^*\right) \tilde{Z}^a \tilde{\partial}^{\cdot b} \end{aligned}$$

is an almost tangent structure which will be called the *almost tangent structure associated to the B<sup>v</sup>-morphism (g, h)*. (See: Definition 4.3)

*Remark 5.3.1* We obtain that

$$\mathcal{J}_{(g, h)}^* \left(\tilde{\delta}_a^*\right) = \mathcal{J}_{(g, h)}^* \left(\tilde{\partial}_a^*\right) = \left(\tilde{g}_{ba} \circ h \circ \pi^*\right) \tilde{\partial}^{\cdot b}$$

and

$$\mathcal{J}_{(g, h)}^* \left(\tilde{\partial}^{\cdot b}\right) = 0.$$

and we have the following properties:

$$(5.3.2) \quad \begin{aligned} \mathcal{J}_{(g, h)}^* \circ \mathcal{P}^* &= \mathcal{J}_{(g, h)}^*; \\ \mathcal{P}^* \circ \mathcal{J}_{(g, h)}^* &= -\mathcal{J}_{(g, h)}^*; \\ \mathcal{J}_{(g, h)}^* \circ \mathcal{H}^* &= \mathcal{J}_{(g, h)}^*; \\ \mathcal{H}^* \circ \mathcal{J}_{(g, h)}^* &= 0; \\ \mathcal{J}_{(g, h)}^* \circ \mathcal{V}^* &= 0; \\ \mathcal{V}^* \circ \mathcal{J}_{(g, h)}^* &= \mathcal{J}_{(g, h)}^*; \\ N_{\mathcal{J}_{(g, h)}^*}^* &= 0. \end{aligned}$$

## 6 Tensor $d$ -fields. Distinguished linear $(\rho, \eta)$ -connections

We consider the following diagram:

$$\begin{array}{ccc} \begin{array}{c} E^* \\ \pi^* \downarrow \\ M \end{array} & \begin{array}{c} (F, [\cdot, \cdot]_{F, h}, (\rho, \eta)) \\ \downarrow \nu \\ N \end{array} \\ & \xrightarrow{h} \end{array}$$

where  $(E, \pi, M) \in |\mathbf{B}^V|$  and  $\left((F, \nu, N), [\cdot, \cdot]_{F,h}, (\rho, \eta)\right)$  is a generalized Lie algebroid.

Let  $(\rho, \eta) \Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ .

Let

$$\left(\mathcal{T}_{q,s}^{p,r} \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right), +, \cdot\right)$$

be the  $\mathcal{F}\left(\overset{*}{E}\right)$ -module of tensor fields by  $(\overset{p,r}{q,s})$ -type from the generalized tangent bundle

$$\left(H(\rho, \eta) T\overset{*}{E} \oplus V(\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right).$$

An arbitrarily tensor field  $T$  is written by the form:

$$T = T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1}^* \otimes \dots \otimes \tilde{\delta}_{\alpha_p}^* \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}^{\cdot b_1} \otimes \dots \otimes \tilde{\partial}^{\cdot b_s} \otimes \delta \tilde{p}_{a_1} \otimes \dots \otimes \delta \tilde{p}_{a_r}.$$

Let

$$\left(\mathcal{T} \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right), +, \cdot, \otimes\right)$$

be the tensor fields algebra of generalized tangent bundle  $\left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$ .

If  $T_1 \in \mathcal{T}_{q_1, s_1}^{p_1, r_1} \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$  and  $T_2 \in \mathcal{T}_{q_2, s_2}^{p_2, r_2} \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$ , then the components of product tensor field  $T_1 \otimes T_2$  are the products of local components of  $T_1$  and  $T_2$ .

Therefore, we obtain  $T_1 \otimes T_2 \in \mathcal{T}_{q_1+q_2, s_1+s_2}^{p_1+p_2, r_1+r_2} \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$ .

Let  $\mathcal{DT} \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$  be the family of tensor fields

$$T \in \mathcal{T} \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

for which there exists

$$T_1 \in \mathcal{T}_{q,0}^{p,0} \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

and

$$T_2 \in \mathcal{T}_{0,s}^{0,r} \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

such that  $T = T_1 + T_2$ .

The  $\mathcal{F}\left(\overset{*}{E}\right)$ -module  $\left(\mathcal{DT} \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right), +, \cdot\right)$  will be called the *module of distinguished tensor fields* or the *module of tensor d-fields*.

*Remark 5.1* The elements of

$$\Gamma \left((\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E}\right)$$

respectively

$$\Gamma((\rho, \eta) T\overset{*}{E})^*, ((\rho, \eta) \tau_{\overset{*}{E}})^*, \overset{*}{E})$$

are tensor  $d$ -fields.

**Definition 6.1** Let  $(\rho, \eta) \Gamma$  be a  $(\rho, \eta)$ -connection for the vector bundle  $\left( \overset{*}{E}, \overset{*}{\pi}, M \right)$  and let

$$(6.4.1) \quad (X, T) \xrightarrow{(\rho, \eta) \overset{*}{D}} (\rho, \eta) \overset{*}{D}_X T$$

be a covariant  $(\rho, \eta)$ -derivative for the tensor algebra of generalized tangent bundle

$$\left( (\rho, \eta) T \overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$$

which preserves the horizontal and vertical distributions by parallelism.

If  $\left( U, \overset{*}{s}_U \right)$  is a vector local  $(m + r)$ -chart for  $\left( \overset{*}{E}, \overset{*}{\pi}, M \right)$ , then the real local functions

$$\left( (\rho, \eta) \overset{*}{H}_{\beta\gamma}, (\rho, \eta) \overset{*}{H}_{b\gamma}, (\rho, \eta) \overset{*}{V}_{\beta}, (\rho, \eta) \overset{*}{V}_a \right)$$

defined on  $\pi^{*-1}(U)$  and determined by the following equalities:

$$(6.2) \quad \begin{aligned} (\rho, \eta) \overset{*}{D}_{\overset{*}{\delta}_\gamma} \overset{*}{\delta}_\beta &= (\rho, \eta) \overset{*}{H}_{\beta\gamma} \overset{*}{\delta}_\alpha, & (\rho, \eta) \overset{*}{D}_{\overset{*}{\delta}_\gamma} \overset{\cdot}{\partial}^a &= (\rho, \eta) \overset{*}{H}_{b\gamma} \overset{\cdot}{\partial}^a \\ (\rho, \eta) \overset{*}{D}_{\overset{\cdot}{\partial}} \overset{*}{\delta}_\beta &= (\rho, \eta) \overset{*}{V}_{\beta} \overset{*}{\delta}_\alpha, & (\rho, \eta) \overset{*}{D}_{\overset{\cdot}{\partial}} \overset{\cdot}{\partial}^b &= (\rho, \eta) \overset{*}{V}_a \overset{\cdot}{\partial}^b \end{aligned}$$

are the components of a linear  $(\rho, \eta)$ -connection

$$\left( (\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V} \right)$$

for the generalized tangent bundle  $\left( (\rho, \eta) T \overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$  which will be called the *distinguished linear  $(\rho, \eta)$ -connection*.

If  $h = Id_M$ , then the distinguished linear  $(Id_{TM}, Id_M)$ -connection will be called the *distinguished linear connection*.

The components of a distinguished linear connection  $\left( \overset{*}{H}, \overset{*}{V} \right)$  will be denoted

$$\left( \overset{*}{H}_{jk}, \overset{*}{H}_{bk}, \overset{*}{V}_j, \overset{*}{V}_a \right).$$

**Theorem 6.1** If  $\left( (\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V} \right)$  is a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle  $\left( (\rho, \eta) T \overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ , then its components satisfy the change relations:

$$\begin{aligned}
(\rho, \eta) \overset{*}{H}_{\beta\gamma}^{\alpha'} &= \Lambda_{\alpha}^{\alpha'} \circ h \circ \overset{*}{\pi} \left[ \Gamma \left( \overset{*}{\rho}, Id_{\overset{*}{E}} \right) \left( \overset{*}{\tilde{\delta}}_{\gamma} \right) \left( \Lambda_{\beta}^{\alpha} \circ h \circ \overset{*}{\pi} \right) + \right. \\
&\quad \left. + (\rho, \eta) \overset{*}{H}_{\beta\gamma}^{\alpha} \cdot \Lambda_{\beta}^{\beta} \circ h \circ \overset{*}{\pi} \right] \cdot \Lambda_{\gamma}^{\gamma} \circ h \circ \overset{*}{\pi}, \\
(\rho, \eta) \overset{*}{H}_{b\gamma}^{a'} &= M_a^{a'} \circ \overset{*}{\pi} \left[ \Gamma \left( \overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}} \right) \left( \overset{*}{\tilde{\delta}}_{\gamma} \right) \left( M_b^a \circ \overset{*}{\pi} \right) + \right. \\
&\quad \left. + (\rho, \eta) \overset{*}{H}_{b\gamma}^a \cdot M_b^b \circ \overset{*}{\pi} \right] \cdot \Lambda_{\gamma}^{\gamma} \circ h \circ \overset{*}{\pi}, \\
(\rho, \eta) \overset{*}{V}_{\beta}^{\alpha c'} &= \Lambda_{\alpha}^{\alpha'} \circ h \circ \overset{*}{\pi} \cdot (\rho, \eta) \overset{*}{V}_{\beta}^{\alpha c} \cdot \Lambda_{\beta}^{\beta} \circ h \circ \overset{*}{\pi} \cdot M_c^c \circ \overset{*}{\pi}, \\
(\rho, \eta) \overset{*}{V}_b^{a c'} &= M_a^{a'} \circ \overset{*}{\pi} \cdot (\rho, \eta) \overset{*}{V}_b^{a c} \cdot M_b^b \circ \overset{*}{\pi} \cdot M_c^c \circ \overset{*}{\pi}.
\end{aligned}
\tag{6.3}$$

The components of a distinguished linear connection  $\left( \overset{*}{H}, \overset{*}{V} \right)$  verify the change relations:

$$\begin{aligned}
\overset{*}{H}_{j\kappa}^{i'} &= \frac{\partial x^{i'}}{\partial x^i} \circ \overset{*}{\pi} \cdot \left[ \frac{\delta}{\delta x^k} \left( \frac{\partial x^i}{\partial x^j} \circ \overset{*}{\pi} \right) + \overset{*}{H}_{jk}^i \cdot \frac{\partial x^j}{\partial x^j} \circ \overset{*}{\pi} \right] \cdot \frac{\partial x^k}{\partial x^k} \circ \overset{*}{\pi}, \\
\overset{*}{H}_{b\kappa}^{a'} &= M_a^{a'} \circ \overset{*}{\pi} \cdot \left[ \frac{\delta}{\delta x^k} \left( M_b^a \circ \overset{*}{\pi} \right) + \overset{*}{H}_{bk}^a \cdot M_b^b \circ \overset{*}{\pi} \right] \cdot \frac{\partial x^k}{\partial x^k} \circ \overset{*}{\pi}, \\
\overset{*}{V}_{j'}^{i c'} &= \frac{\partial x^{i'}}{\partial x^i} \circ \overset{*}{\pi} \cdot \overset{*}{V}_j^{i c} \frac{\partial x^j}{\partial x^j} \circ \overset{*}{\pi} \cdot M_c^c \circ \overset{*}{\pi}, \\
\overset{*}{V}_{b'}^{a c'} &= M_a^{a'} \circ \overset{*}{\pi} \cdot \overset{*}{V}_b^{a c} \cdot M_b^b \circ \overset{*}{\pi} \cdot M_c^c \circ \overset{*}{\pi}.
\end{aligned}
\tag{6.3'}$$

**Example 6.1** If  $\left( \overset{*}{E}, \overset{*}{\pi}, M \right)$  is endowed with the  $(\rho, \eta)$ -connection  $(\rho, \eta) \Gamma$ , then the local real functions

$$\left( \frac{\partial(\rho, \eta) \Gamma_{b\gamma}}{\partial p_a}, \frac{\partial(\rho, \eta) \Gamma_{b\gamma}}{\partial p_a}, 0, 0 \right)
\tag{6.4}$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle

$$\left( (\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right),$$

which will be called the *Berwald linear  $(\rho, \eta)$ -connection*.

**Theorem 6.2** If the generalized tangent bundle  $\left( (\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$  is endowed with a distinguished linear  $(\rho, \eta)$ -connection  $((\rho, \eta) \overset{*}{H}, (\rho, \eta) \overset{*}{V})$ , then, for any

$$X = \tilde{Z}^{\gamma} \tilde{\delta}_{\gamma}^* + Y_a^{\cdot a} \tilde{\partial}^a \in \Gamma \left( (\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$$

and for any

$$T \in \mathcal{T}_{qs}^{pr} \left( (\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right),$$

we obtain the formula:

$$\begin{aligned}
(\rho, \eta) D_X \left( T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1}^* \otimes \dots \otimes \tilde{\delta}_{\alpha_p}^* \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes \right. \\
\left. \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}^{\cdot b_1} \otimes \dots \otimes \tilde{\partial}^{\cdot b_s} \otimes \delta \tilde{p}_{a_1} \otimes \dots \otimes \delta \tilde{p}_{a_r} \right) = \\
= \tilde{Z}^\gamma T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \tilde{\delta}_{\alpha_1}^* \otimes \dots \otimes \tilde{\delta}_{\alpha_p}^* \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}^{\cdot b_1} \otimes \dots \otimes \\
\otimes \tilde{\partial}^{\cdot b_s} \otimes \delta \tilde{p}_{a_1} \otimes \dots \otimes \delta \tilde{p}_{a_r} + Y_c T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |^c \tilde{\delta}_{\alpha_1}^* \otimes \dots \otimes \\
\otimes \tilde{\delta}_{\alpha_p}^* \otimes d\tilde{z}^{\beta_1} \otimes \dots \otimes d\tilde{z}^{\beta_q} \otimes \tilde{\partial}^{\cdot b_1} \otimes \dots \otimes \tilde{\partial}^{\cdot b_s} \otimes \delta \tilde{p}_{a_1} \otimes \dots \otimes \delta \tilde{p}_{a_r},
\end{aligned}$$

where

$$\begin{aligned}
T_{\beta_1 \dots \beta_q b_1 \dots b_s | \gamma}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} &= \Gamma \left( \tilde{\rho}, Id_E^* \right) \left( \tilde{\delta}_\gamma^* \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
&+ (\rho, \eta) H_{\alpha \gamma}^* T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) H_{\alpha \gamma}^* T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_{p-1} \alpha a_1 \dots a_r} \\
&- (\rho, \eta) H_{\beta_1 \gamma}^* T_{\beta \beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) H_{\beta_q \gamma}^* T_{\beta_1 \dots \beta_{q-1} \beta b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
&- (\rho, \eta) H_{a \gamma}^* T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a a_2 \dots a_r} - \dots - (\rho, \eta) H_{a \gamma}^* T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\
&+ (\rho, \eta) H_{b_1 \gamma}^* T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) H_{b_s \gamma}^* T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}
\end{aligned}$$

and

$$\begin{aligned}
T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} |^c &= \Gamma \left( \tilde{\rho}, Id_E^* \right) \left( \tilde{\partial}^{\cdot c} \right) T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} + \\
&+ (\rho, \eta) V_\alpha^* T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha \alpha_2 \dots \alpha_p a_1 \dots a_r} + \dots + (\rho, \eta) V_\alpha^* T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha \dots \alpha_{p-1} \alpha a_1 \dots a_r} \\
&- (\rho, \eta) V_{\beta_1}^* T_{\beta \beta_2 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} - \dots - (\rho, \eta) V_{\beta_q}^* T_{\beta_1 \dots \beta_{q-1} \beta b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \\
&- (\rho, \eta) V_a^* T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a a_2 \dots a_r} - \dots - (\rho, \eta) V_a^* T_{\beta_1 \dots \beta_q b_1 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_{r-1} a} \\
&+ (\rho, \eta) V_{b_1}^* T_{\beta_1 \dots \beta_q b b_2 \dots b_s}^{\alpha_1 \dots \alpha_p a_1 \dots a_r} \dots + (\rho, \eta) V_{b_s}^* T_{\beta_1 \dots \beta_q b_1 \dots b_{s-1} b}^{\alpha_1 \dots \alpha_p a_1 \dots a_r}.
\end{aligned}$$

**Definition 6.2** We assume that  $(E, \pi, M) = (F, \nu, N)$ .

If  $(\rho, \eta) \Gamma$  is a  $(\rho, \eta)$ -connection for the vector bundle  $\left( \tilde{E}, \tilde{\pi}, M \right)$  and

$$\left( (\rho, \eta) H_{bc}^* \tilde{H}_{bc}^*, (\rho, \eta) V_b^* \tilde{V}_b^* \right)$$

are the components of a distinguished linear  $(\rho, \eta)$ -connection for the generalized tangent bundle  $\left( (\rho, \eta) T\tilde{E}, (\rho, \eta) \tau_{\tilde{E}}^*, \tilde{E} \right)$  such that

$$(\rho, \eta) H_{bc}^* = (\rho, \eta) \tilde{H}_{bc}^* \text{ and } (\rho, \eta) V_b^* = (\rho, \eta) \tilde{V}_b^*,$$

then we will say that *the generalized tangent bundle*  $\left((\rho, \eta) T^*E, (\rho, \eta) \tau_E^*, E\right)$  *is endowed with a normal distinguished linear*  $(\rho, \eta)$ -*connection on components*

$$\left((\rho, \eta) \overset{*}{H}_{bc}^a, (\rho, \eta) \overset{*}{V}_b^{ac}\right).$$

The components of a normal distinguished linear  $(Id_{TM}, Id_M)$ -connection  $\left(\overset{*}{H}, \overset{*}{V}\right)$  will be denoted  $\left(\overset{*}{H}_{jk}^i, \overset{*}{V}_{jk}^i\right)$ .

## 7 Dual mechanical systems

Using the diagram:

$$(7.1) \quad \begin{array}{ccc} \overset{*}{E} & & (E, [\cdot, \cdot]_{E,h}, (\rho, \eta)) \\ \overset{*}{\pi} \downarrow & & \downarrow \pi \\ M & \xrightarrow{h} & M \end{array}$$

where  $\left((E, \pi, M), [\cdot, \cdot]_{E,h}, (\rho, \eta)\right)$  is a generalized Lie algebroid, we build the generalized tangent bundle

$$(((\rho, \eta) T^*E, (\rho, \eta) \tau_E^*, E), [\cdot, \cdot]_{(\rho, \eta) T^*E}, (\tilde{\rho}, Id_E^*)).$$

**Definition 7.1** A triple

$$(7.2) \quad \left(\left(\overset{*}{E}, \overset{*}{\pi}, M\right), \overset{*}{F}_e, (\rho, \eta) \Gamma\right),$$

where

$$(7.3) \quad \overset{*}{F}_e = F_a \overset{\cdot}{\partial}^a \in \Gamma\left(V(\rho, \eta) T^*E, (\rho, \eta) \tau_E^*, E\right)$$

is an external force and  $(\rho, \eta) \Gamma$  is a  $(\rho, \eta)$ -connection for  $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ , will be called *dual mechanical*  $(\rho, \eta)$ -*system*.

**Definition 7.2** A smooth *Hamilton fundamental function* on the dual vector bundle  $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$  is a mapping  $\overset{*}{E} \xrightarrow{H} \mathbb{R}$  which satisfies the following conditions:

1.  $H \circ \overset{*}{u} \in C^\infty(M)$ , for any  $\overset{*}{u} \in \Gamma\left(\overset{*}{E}, \overset{*}{\pi}, M\right) \setminus \{0\}$ ;
2.  $H \circ 0 \in C^0(M)$ , where 0 means the null section of  $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ .

Let  $H$  be a differentiable Hamiltonian on the total space of the dual vector bundle  $\left(\overset{*}{E}, \overset{*}{\pi}, M\right)$ .

If  $(U, s_U)$  is a local vector  $(m+r)$ -chart for  $(E, \pi, M)$ , then we obtain the following real functions defined on  $\pi^{*-1}(U)$ :

$$(7.4) \quad \begin{aligned} H_i &\stackrel{\text{put}}{=} \frac{\partial H}{\partial x^i} \stackrel{\text{put}}{=} \frac{\partial}{\partial x^i} (H) & H_i^b &\stackrel{\text{put}}{=} \frac{\partial^2 H}{\partial x^i \partial p_b} \stackrel{\text{put}}{=} \frac{\partial}{\partial x^i} \left( \frac{\partial}{\partial p_b} (H) \right) \\ H^a &\stackrel{\text{put}}{=} \frac{\partial H}{\partial p_a} \stackrel{\text{put}}{=} \frac{\partial}{\partial p_a} (H) & H^{ab} &\stackrel{\text{put}}{=} \frac{\partial^2 H}{\partial p_a \partial p_b} \stackrel{\text{put}}{=} \frac{\partial}{\partial p_a} \left( \frac{\partial}{\partial p_b} (H) \right). \end{aligned}$$

**Definition 7.3** If for any local vector  $m+r$ -chart  $(U, s_U)$  of  $(E, \pi, M)$ , we have:

$$(7.5) \quad \text{rank} \left\| H^{ab} \left( u_x^* \right) \right\| = r,$$

for any  $u_x^* \in \pi^{*-1}(U) \setminus \{0_x\}$ , then we say that *the Hamiltonian  $H$  is regular*.

**Proposition 7.1** If the Hamiltonian  $H$  is regular, then for any local vector  $m+r$ -chart  $(U, s_U)$  of  $(E, \pi, M)$ , we obtain the real functions  $\tilde{H}_{ba}$  locally defined by

$$(7.6) \quad \begin{array}{ccc} \pi^{*-1}(U) & \xrightarrow{\tilde{H}_{ba}} & \mathbb{R} \\ u_x^* & \longmapsto & \tilde{H}_{ba} \left( u_x^* \right) \end{array}$$

where  $\left\| \tilde{H}_{ba} \left( u_x^* \right) \right\| = \left\| H^{ab} \left( u_x^* \right) \right\|^{-1}$ , for any  $u_x^* \in \pi^{*-1}(U) \setminus \{0_x\}$ .

**Definition 7.4** A smooth *Cartan fundamental function* on the vector bundle  $(E, \pi, M)$

is a mapping  $E \xrightarrow{K} \mathbb{R}_+$  which satisfies the following conditions:

1.  $K \circ u^* \in C^\infty(M)$ , for any  $u^* \in \Gamma(E, \pi, M) \setminus \{0\}$ ;
2.  $K \circ 0 \in C^0(M)$ , where  $0$  means the null section of  $(E, \pi, M)$ ;
3.  $K$  is positively 1-homogenous on the fibres of vector bundle  $(E, \pi, M)$ ;
4. For any local vector  $m+r$ -chart  $(U, s_U)$  of  $(E, \pi, M)$ , the hessian:

$$(7.7) \quad \left\| K^{2 \ ab} \left( u_x^* \right) \right\|$$

is positively define for any  $u_x^* \in \pi^{*-1}(U) \setminus \{0_x\}$ .

**Definition 7.5** If  $H$  respectively  $K$  is a smooth Hamilton respectively Cartan function, then we put the triple

$$\left( \left( E, \pi, M \right), F_e, H \right),$$

respectively

$$\left( \left( E, \pi, M \right), F_e, K \right),$$

where

$$\overset{*}{F}_e = F_a \overset{\cdot}{\tilde{\partial}}^a \in \Gamma \left( V(\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$$

is an external force. These are called *Hamilton mechanical*  $(\rho, \eta)$ -system and *Cartan mechanical*  $(\rho, \eta)$ -system respectively.

Any Hamilton mechanical  $(Id_{TM}, Id_M)$ -system and any Cartan mechanical  $(Id_{TM}, Id_M)$ -system will be called *Hamilton mechanical system* and *Cartan mechanical system*, respectively.

## 8 $(\rho, \eta)$ -semisprays and $(\rho, \eta)$ -sprays for dual mechanical $(\rho, \eta)$ -systems

Let  $\left( \left( \overset{*}{E}, \overset{*}{\pi}, M \right), \overset{*}{F}_e, (\rho, \eta) \Gamma \right)$  be an arbitrary dual mechanical  $(\rho, \eta)$ -system.

**Definition 8.1** The vertical section  $\overset{*}{\mathbb{C}} = p_a \overset{\cdot}{\tilde{\partial}}^a$  will be called the *Liouville section*.

A section  $\overset{*}{S} \in \Gamma \left( (\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$  will be called  $(\rho, \eta)$ -semispray if there exists an almost tangent structure  $e$  such that  $e \left( \overset{*}{S} \right) = \overset{*}{\mathbb{C}}$ .

Let  $g \in \mathbf{Man} \left( \overset{*}{E}, E \right)$  be such that  $(g, h)$  is a locally invertible  $\mathbf{B}^v$ -morphism of  $\left( \overset{*}{E}, \overset{*}{\pi}, M \right)$  source and  $(E, \pi, M)$  target.

**Theorem 8.1** *The section*

$$(8.1) \quad \overset{*}{S} = \left( g^{ab} \circ h \circ \overset{*}{\pi} \right) p_b \overset{*}{\tilde{\partial}}_a - 2 \left( G_a - \frac{1}{4} F_a \right) \overset{\cdot}{\tilde{\partial}}^a$$

is a  $(\rho, \eta)$ -semispray such that the real local functions  $G_a$ ,  $a \in \overline{1, n}$ , satisfy the following conditions

$$(8.2) \quad \begin{aligned} (\rho, \eta) \Gamma_{bc} &= \left( \tilde{g}_{ca} \circ h \circ \overset{*}{\pi} \right) \frac{\partial (G_b - \frac{1}{4} F_b)}{\partial p_a} \\ &- \frac{1}{2} \left( g^{de} \circ h \circ \overset{*}{\pi} \right) p_e \left( L_{dc}^f \circ h \circ \overset{*}{\pi} \right) \left( \tilde{g}_{fb} \circ h \circ \overset{*}{\pi} \right) \\ &+ \frac{1}{2} \left( \rho_c^j \circ h \circ \overset{*}{\pi} \right) \frac{\partial (g^{ae} \circ h \circ \overset{*}{\pi})}{\partial x^j} p_e \left( \tilde{g}_{ab} \circ h \circ \overset{*}{\pi} \right) \\ &- \frac{1}{2} \left( g^{ae} \circ h \circ \overset{*}{\pi} \right) p_e \left( \rho_b^i \circ h \circ \overset{*}{\pi} \right) \frac{\partial (\tilde{g}_{ac} \circ h \circ \overset{*}{\pi})}{\partial x^i} \end{aligned}$$

In addition, we remark that the local real functions

$$(8.3) \quad \begin{aligned} (\rho, \eta) \overset{\circ}{\Gamma}_{bc} &= \left( \tilde{g}_{ca} \circ h \circ \overset{*}{\pi} \right) \frac{\partial G_b}{\partial p_a} \\ &- \frac{1}{2} \left( g^{de} \circ h \circ \overset{*}{\pi} \right) p_e \left( L_{dc}^f \circ h \circ \overset{*}{\pi} \right) \left( \tilde{g}_{fb} \circ h \circ \overset{*}{\pi} \right) \\ &+ \frac{1}{2} \left( \rho_c^j \circ h \circ \overset{*}{\pi} \right) \frac{\partial (g^{ae} \circ h \circ \overset{*}{\pi})}{\partial x^j} p_e \left( \tilde{g}_{ab} \circ h \circ \overset{*}{\pi} \right) \\ &- \frac{1}{2} \left( g^{ae} \circ h \circ \overset{*}{\pi} \right) p_e \left( \rho_b^i \circ h \circ \overset{*}{\pi} \right) \frac{\partial (\tilde{g}_{ac} \circ h \circ \overset{*}{\pi})}{\partial x^i} \end{aligned}$$



are the components of a  $(\rho, \eta)$ -connection  $(\rho, \eta) \overset{*}{\Gamma}$  for the vector bundle  $\left( \overset{*}{E}, \overset{*}{\pi}, M \right)$ .

The  $(\rho, \eta)$ -semispray  $\overset{*}{S}$  will be called *the canonical  $(\rho, \eta)$ -semispray associated to mechanical  $(\rho, \eta)$ -system*  $\left( \left( \overset{*}{E}, \overset{*}{\pi}, M \right), \overset{*}{F}_e, (\rho, \eta) \overset{*}{\Gamma} \right)$  and from locally invertible  $\mathbf{B}^v$ -morphism  $(g, h)$ .

*Proof.* We consider the **Mod**-endomorphism

$$\begin{aligned} \Gamma \left( (\rho, \eta) TE, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right) &\xrightarrow{\mathbb{P}} \Gamma \left( (\rho, \eta) TE, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right) \\ X &\longmapsto \overset{*}{J}_{(g, h)} \left[ \overset{*}{S}, X \right]_{(\rho, \eta) TE} - \left[ \overset{*}{S}, \overset{*}{J}_{(g, h)} X \right]_{(\rho, \eta) TE}. \end{aligned}$$

Let  $X = Z^a \overset{*}{\tilde{\partial}}_a + Y_a \overset{\cdot}{\tilde{\partial}}^a$  be an arbitrary section. Since

$$\begin{aligned} \left[ \overset{*}{S}, X \right]_{(\rho, \eta) TE} &= \left[ \left( g^{ae} \circ h \circ \overset{*}{\pi} \right) p_e \overset{*}{\tilde{\partial}}_a, Z^b \overset{*}{\tilde{\partial}}_b \right]_{(\rho, \eta) TE} + \left[ \left( g^{ae} \circ h \circ \overset{*}{\pi} \right) p_e \overset{*}{\tilde{\partial}}_a, Y_b \overset{\cdot}{\tilde{\partial}}^b \right]_{(\rho, \eta) TE} \\ &\quad - \left[ 2 \left( G_a - \frac{1}{4} F_a \right) \overset{\cdot}{\tilde{\partial}}^a, Z^b \overset{*}{\tilde{\partial}}_b \right]_{(\rho, \eta) TE} - \left[ 2 \left( G_a - \frac{1}{4} F_a \right) \overset{\cdot}{\tilde{\partial}}^a, Y_b \overset{\cdot}{\tilde{\partial}}^b \right]_{(\rho, \eta) TE} \end{aligned}$$

and

$$\begin{aligned} \left[ \left( g^{ae} \circ h \circ \overset{*}{\pi} \right) p_e \overset{*}{\tilde{\partial}}_a, Z^b \overset{*}{\tilde{\partial}}_b \right]_{(\rho, \eta) TE} &= \left( g^{ae} \circ h \circ \overset{*}{\pi} \right) p_e \left( \rho_a^i \circ h \circ \overset{*}{\pi} \right) \frac{\partial Z^c}{\partial x^i} \overset{*}{\tilde{\partial}}_c \\ &\quad - Z^b \left( \rho_b^j \circ h \circ \overset{*}{\pi} \right) \frac{\partial \left( g^{ce} \circ h \circ \overset{*}{\pi} \right)}{\partial x^j} p_e \overset{*}{\tilde{\partial}}_c \\ &\quad + \left( g^{ae} \circ h \circ \overset{*}{\pi} \right) p_e Z^b \left( L_{ab}^c \circ h \circ \overset{*}{\pi} \right) \overset{*}{\tilde{\partial}}_c, \\ \left[ \left( g^{ae} \circ h \circ \overset{*}{\pi} \right) p_e \overset{*}{\tilde{\partial}}_a, Y_b \overset{\cdot}{\tilde{\partial}}^b \right]_{(\rho, \eta) TE} &= \left( g^{ae} \circ h \circ \overset{*}{\pi} \right) p_e \left( \rho_a^i \circ h \circ \overset{*}{\pi} \right) \frac{\partial Y_c}{\partial x^i} \overset{\cdot}{\tilde{\partial}}^c \\ &\quad - Y_b \left( g^{bc} \circ h \circ \overset{*}{\pi} \right) \overset{*}{\tilde{\partial}}_c, \\ \left[ 2 \left( G_a - \frac{1}{4} F_a \right) \overset{\cdot}{\tilde{\partial}}^a, Z^b \overset{*}{\tilde{\partial}}_b \right]_{(\rho, \eta) TE} &= 2 \left( G_a - \frac{1}{4} F_a \right) \frac{\partial Z^c}{\partial p_a} \overset{*}{\tilde{\partial}}_c \\ &\quad - 2 Z^b \left( \rho_b^j \circ h \circ \overset{*}{\pi} \right) \frac{\partial \left( G_c - \frac{1}{4} F_c \right)}{\partial x^j} \overset{\cdot}{\tilde{\partial}}^c, \\ \left[ 2 \left( G_a - \frac{1}{4} F_a \right) \overset{\cdot}{\tilde{\partial}}^a, Y_b \overset{\cdot}{\tilde{\partial}}^b \right]_{(\rho, \eta) TE} &= 2 \left( G_a - \frac{1}{4} F_a \right) \frac{\partial Y_c}{\partial y^a} \overset{\cdot}{\tilde{\partial}}^c - 2 Y_b \frac{\partial \left( G_c - \frac{1}{4} F_c \right)}{\partial p_b} \overset{\cdot}{\tilde{\partial}}^c, \end{aligned}$$

it results that

$$\begin{aligned}
(P_1) \quad \mathcal{J}_{(g,h)}^* \left[ S, X \right]_{(\rho,\eta)TE}^* &= \left( g^{ae} \circ h \circ \pi^* \right) p_e \left( \rho_a^i \circ h \circ \pi^* \right) \frac{\partial Z^c}{\partial x^i} \left( \tilde{g}_{cd} \circ h \circ \pi^* \right) \dot{\tilde{\partial}}^d \\
&\quad - Z^b \left( \rho_b^j \circ h \circ \pi^* \right) \frac{\partial \left( g^{ce} \circ h \circ \pi^* \right)}{\partial x^j} p_e \left( \tilde{g}_{cd} \circ h \circ \pi^* \right) \dot{\tilde{\partial}}^d \\
&\quad + \left( g^{ae} \circ h \circ \pi^* \right) p_e Z^b \left( L_{ab}^c \circ h \circ \pi^* \right) \dot{\tilde{\partial}}^d - Y_d \dot{\tilde{\partial}}^d \\
&\quad - 2 \left( G_a - \frac{1}{4} F_a \right) \frac{\partial Z^c}{\partial p_a} \left( \tilde{g}_{cd} \circ h \circ \pi^* \right) \dot{\tilde{\partial}}^d.
\end{aligned}$$

Since

$$\begin{aligned}
\left[ S, \mathcal{J}_{(g,h)}^* X \right]_{(\rho,\eta)TE}^* &= \left[ \left( g^{ae} \circ h \circ \pi^* \right) p_e \dot{\tilde{\partial}}_a^*, Z^b \left( \tilde{g}_{bc} \circ h \circ \pi^* \right) \dot{\tilde{\partial}}^c \right]_{(\rho,\eta)TE}^* \\
&\quad - \left[ 2 \left( G_a - \frac{1}{4} F_a \right) \dot{\tilde{\partial}}^a, Z^b \left( \tilde{g}_{bc} \circ h \circ \pi^* \right) \dot{\tilde{\partial}}^c \right]_{(\rho,\eta)TE}^*
\end{aligned}$$

and

$$\begin{aligned}
\left[ \left( g^{ae} \circ h \circ \pi^* \right) p_e \dot{\tilde{\partial}}_a^*, Z^b \left( \tilde{g}_{bc} \circ h \circ \pi^* \right) \dot{\tilde{\partial}}^c \right]_{(\rho,\eta)TE}^* &= -Z^d \dot{\tilde{\partial}}_d^* \\
&\quad + \left( g^{ae} \circ h \circ \pi^* \right) p_e \left( \rho_a^i \circ h \circ \pi^* \right) \frac{\partial Z^b}{\partial x^i} \left( \tilde{g}_{bd} \circ h \circ \pi^* \right) \dot{\tilde{\partial}}^d \\
&\quad - \left( g^{ae} \circ h \circ \pi^* \right) p_e \left( \rho_a^i \circ h \circ \pi^* \right) Z^b \frac{\partial \left( \tilde{g}_{bd} \circ h \circ \pi^* \right)}{\partial x^i} \dot{\tilde{\partial}}^d \\
\left[ 2 \left( G_a - \frac{1}{4} F_a \right) \dot{\tilde{\partial}}^a, Z^b \left( \tilde{g}_{bc} \circ h \circ \pi^* \right) \dot{\tilde{\partial}}^c \right]_{(\rho,\eta)TE}^* &= 2 \left( G_a - \frac{1}{4} F_a \right) \frac{\partial Z^b}{\partial p_a} \left( \tilde{g}_{bd} \circ h \circ \pi^* \right) \dot{\tilde{\partial}}^d \\
&\quad - Z^b \left( \tilde{g}_{bc} \circ h \circ \pi^* \right) \frac{\partial 2 \left( G_d - \frac{1}{4} F_d \right)}{\partial p_c} \dot{\tilde{\partial}}^d
\end{aligned}$$

it results that

$$\begin{aligned}
(P_2) \quad \left[ S, \mathcal{J}_{(g,h)}^* X \right]_{(\rho,\eta)TE}^* &= -Z^d \dot{\tilde{\partial}}_d^* + \left( g^{ae} \circ h \circ \pi^* \right) p_e \left( \rho_a^i \circ h \circ \pi^* \right) \frac{\partial Z^b}{\partial x^i} \left( \tilde{g}_{bd} \circ h \circ \pi^* \right) \dot{\tilde{\partial}}^d \\
&\quad - \left( g^{ae} \circ h \circ \pi^* \right) p_e \left( \rho_a^i \circ h \circ \pi^* \right) Z^b \frac{\partial \left( \tilde{g}_{bd} \circ h \circ \pi^* \right)}{\partial x^i} \dot{\tilde{\partial}}^d \\
&\quad - 2 \left( G_a - \frac{1}{4} F_a \right) \frac{\partial Z^b}{\partial p_a} \left( \tilde{g}_{bd} \circ h \circ \pi^* \right) \dot{\tilde{\partial}}^d \\
&\quad + Z^b \left( \tilde{g}_{bc} \circ h \circ \pi^* \right) \frac{\partial 2 \left( G_d - \frac{1}{4} F_d \right)}{\partial p_c} \dot{\tilde{\partial}}^d.
\end{aligned}$$

Using equalities  $(P_1)$  and  $(P_2)$ , we obtain:

$$\begin{aligned} \mathbb{P} \left( Z^a \tilde{\partial}_a + Y \tilde{\partial}^{\cdot a} \right) &= Z^a \tilde{\partial}_a^* - Y_d \tilde{\partial}^{\cdot d} + \left( g^{ae} \circ h \circ \pi^* \right) p_e Z^b \left( L_{ab}^c \circ h \circ \pi^* \right) \left( \tilde{g}_{cd} \circ h \circ \pi^* \right) \tilde{\partial}^{\cdot d} \\ &\quad - Z^b \left( \rho_b^j \circ h \circ \pi^* \right) \frac{\partial (g^{ce} \circ h \circ \pi^*)}{\partial x^j} p_e \left( \tilde{g}_{cd} \circ h \circ \pi^* \right) \tilde{\partial}^{\cdot d} \\ &\quad + \left( g^{ae} \circ h \circ \pi^* \right) p_e \left( \rho_a^i \circ h \circ \pi^* \right) Z^b \frac{\partial (\tilde{g}_{bd} \circ h \circ \pi^*)}{\partial x^i} \tilde{\partial}^{\cdot d} \\ &\quad - Z^b \left( \tilde{g}_{bc} \circ h \circ \pi^* \right) \frac{\partial (G_d - \frac{1}{4} F_d)}{\partial p_c} \tilde{\partial}^{\cdot d} \end{aligned}$$

After some calculations, it results that  $\mathbb{P}$  is an almost product structure.

Using the equalities (5.1.2) and (5.2.2) it results that

$$\mathbb{P} \left( Z^a \tilde{\partial}_a + Y_a \tilde{\partial}^{\cdot a} \right) = (Id - 2(\rho, \eta) \Gamma) \left( Z^a \tilde{\partial}_a + Y_a \tilde{\partial}^{\cdot a} \right),$$

for any  $Z^a \tilde{\partial}_a + Y \tilde{\partial}^{\cdot a} \in \Gamma \left( (\rho, \eta) T E, (\rho, \eta) \tau_E^*, E \right)$  and we obtain

$$\begin{aligned} (\rho, \eta) \Gamma \left( Z^a \tilde{\partial}_a + Y \tilde{\partial}^{\cdot a} \right) &= Y_d \tilde{\partial}^{\cdot d} - \frac{1}{2} \left( g^{ae} \circ h \circ \pi^* \right) p_e Z^b \left( L_{ab}^c \circ h \circ \pi^* \right) \left( \tilde{g}_{cd} \circ h \circ \pi^* \right) \tilde{\partial}^{\cdot d} \\ &\quad + \frac{1}{2} Z^b \left( \rho_b^j \circ h \circ \pi^* \right) \frac{\partial (g^{ce} \circ h \circ \pi^*)}{\partial x^j} p_e \left( \tilde{g}_{cd} \circ h \circ \pi^* \right) \tilde{\partial}^{\cdot d} \\ &\quad - \frac{1}{2} \left( g^{ae} \circ h \circ \pi^* \right) p_e \left( \rho_a^i \circ h \circ \pi^* \right) Z^b \frac{\partial (\tilde{g}_{bd} \circ h \circ \pi^*)}{\partial x^i} \tilde{\partial}^{\cdot d} \\ &\quad + Z^b \left( \tilde{g}_{bc} \circ h \circ \pi^* \right) \frac{\partial (G_d - \frac{1}{4} F_d)}{\partial p_c} \tilde{\partial}^{\cdot d}. \end{aligned}$$

Since

$$(\rho, \eta) \Gamma \left( Z^a \tilde{\partial}_a + Y_a \tilde{\partial}^{\cdot a} \right) = (Y_d + (\rho, \eta) \Gamma_{db} Z^b) \tilde{\partial}^{\cdot d}$$

it results the relations (8.3). In addition, since

$$(\rho, \eta) \mathring{\Gamma}_{bc} = (\rho, \eta) \Gamma_{bc} + \frac{1}{4} \left( \tilde{g}_{cd} \circ h \circ \pi^* \right) \frac{\partial F_b}{\partial p_d}$$

and

$$\begin{aligned} (\rho, \eta) \mathring{\Gamma}_{b'c'} &= (\rho, \eta) \Gamma_{b'c'} - \frac{1}{4} \left( \tilde{g}_{c'e} \circ h \circ \pi^* \right) \frac{\partial F_{b'}}{\partial p_{e'}} \\ &= M_{b'}^b \circ \pi^* \left( - \left( \rho_c^i \circ h \circ \pi^* \right) \frac{\partial M_{b'}^{a'}}{\partial x^i} p_{a'} + (\rho, \eta) \Gamma_{bc} \right) M_{c'}^c \circ h \circ \pi^* \\ &\quad + M_{b'}^b \circ \pi^* \left( \frac{1}{4} \left( \tilde{g}_{ce} \circ h \circ \pi^* \right) \frac{\partial F_b}{\partial p_e} \right) M_{c'}^c \circ h \circ \pi^* \\ &= M_{b'}^b \circ \pi^* \left( - \left( \rho_c^i \circ h \circ \pi^* \right) \frac{\partial M_{b'}^{a'}}{\partial x^i} p_{a'} + \left( (\rho, \eta) \Gamma_{bc} - \frac{1}{4} \left( \tilde{g}_{ce} \circ h \circ \pi^* \right) \cdot \frac{\partial F_b}{\partial p_e} \right) \right) M_{c'}^c \circ h \circ \pi^* \\ &= M_{b'}^b \circ \pi^* \left( - \left( \rho_c^i \circ h \circ \pi^* \right) \frac{\partial M_{b'}^{a'}}{\partial x^i} p_{a'} + (\rho, \eta) \mathring{\Gamma}_{bc} \right) M_{c'}^c \circ h \circ \pi^* \end{aligned}$$

it results the conclusion of the theorem.

*q.e.d.*

*Remark 8.1* In particular, if  $(\rho, \eta) = (Id_{TM}, Id_M)$ ,  $(g, h) = (Id_E, Id_M)$ , and  $F_e = 0$ , then we obtain the classical canonical semispray associated to connection  $\Gamma$ .

Using *Theorem 8.1*, we obtain the following:

**Theorem 8.2** *The following properties hold good:*

1° Since  $\overset{*}{\underset{\circ}{\delta}}_c = \overset{*}{\delta}_c + (\rho, \eta) \overset{\cdot}{\Gamma}_{bc} \overset{\cdot}{\delta}^b$ ,  $c \in \overline{1, r}$ , it results that

$$(8.4) \quad \overset{*}{\underset{\circ}{\delta}}_c = \overset{*}{\delta}_c - \frac{1}{4} \left( \tilde{g}_{ce} \circ h \circ \pi^* \right) \frac{\partial F_b}{\partial p_e} \overset{\cdot}{\delta}^b, \quad c \in \overline{1, r}.$$

2° Since  $\overset{*}{\delta} \tilde{p}_b = -(\rho, \eta) \overset{*}{\Gamma}_{bc} d\tilde{z}^c + d\tilde{p}_b$ ,  $b \in \overline{1, r}$ , it results that

$$(8.5) \quad \overset{*}{\delta} \tilde{p}_b = \tilde{p}_b + \frac{1}{4} \left( \tilde{g}_{ec} \circ h \circ \pi^* \right) \frac{\partial F_b}{\partial p_e} d\tilde{z}^c, \quad b \in \overline{1, r}.$$

**Theorem 8.3** *The real local functions*

$$(8.6) \quad \left( \frac{\partial(\rho, \eta) \Gamma_{bc}}{\partial p_a}, \frac{\partial(\rho, \eta) \tilde{\Gamma}_{bc}}{\partial p_a}, 0, 0 \right), \quad a, b, c \in \overline{1, r}$$

and

$$(8.6)' \quad \left( \frac{\partial(\rho, \eta) \overset{\cdot}{\Gamma}_{bc}}{\partial p_a}, \frac{\partial(\rho, \eta) \overset{\cdot}{\tilde{\Gamma}}_{bc}}{\partial p_a}, 0, 0 \right), \quad a, b, c \in \overline{1, r}$$

respectively, are the coefficients to a normal Berwald linear  $(\rho, \eta)$ -connection for the generalized tangent bundle  $\left( (\rho, \eta) TE^*, (\rho, \eta) \tau_E^*, E \right)$ .

**Theorem 8.4** *The tensor of integrability of the  $(\rho, \eta)$ -connection  $(\rho, \eta) \overset{\circ}{\Gamma}$  is as follows:*

$$(8.7) \quad \begin{aligned} (\rho, \eta, h) \overset{\circ}{\mathbb{R}}_{bcd} &= (\rho, \eta, h) \mathbb{R}_{bcd} + \frac{1}{4} \left( \left( \tilde{g}_{de} \circ h \circ \pi^* \right) \frac{\partial F_b}{\partial p_e} \Big|_c - \left( \tilde{g}_{ce} \circ h \circ \pi^* \right) \frac{\partial F_b}{\partial p_e} \Big|_d \right) + \\ &+ \frac{1}{16} \left( \left( \tilde{g}_{ed} \circ h \circ \pi^* \right) \frac{\partial F_l}{\partial p_e} \left( \tilde{g}_{cf} \circ h \circ \pi^* \right) \frac{\partial^2 F_b}{\partial p_l \partial p_f} - \left( \tilde{g}_{cf} \circ h \circ \pi^* \right) \frac{\partial F_l}{\partial p_f} \left( \tilde{g}_{de} \circ h \circ \pi^* \right) \frac{\partial^2 F_b}{\partial p_l \partial p_e} \right) + \\ &+ \frac{1}{4} \left( L_{cd}^f \circ h \circ \pi^* \right) \left( \tilde{g}_{fe} \circ h \circ \pi^* \right) \frac{\partial F_b}{\partial p_e}, \end{aligned}$$

where  $|_c$  is the  $h$ -covariant derivation with respect to the normal Berwald linear  $(\rho, \eta)$ -connection (8.6).

*Proof.* Since

$$\begin{aligned} (\rho, \eta, h) \overset{\circ}{\mathbb{R}}_{bcd} &= \Gamma \left( \overset{*}{\tilde{\rho}}, Id_E^* \right) \left( \overset{\circ}{\delta}_c \right) \left( (\rho, \eta) \overset{\circ}{\Gamma}_{bd} \right) - \Gamma \left( \overset{*}{\tilde{\rho}}, Id_E^* \right) \left( \overset{\circ}{\delta}_d \right) \left( (\rho, \eta) \overset{\circ}{\Gamma}_{bc} \right) \\ &- \left( L_{cd}^e \circ h \circ \pi^* \right) (\rho, \eta) \overset{\circ}{\Gamma}_{be}, \end{aligned}$$

and

$$\begin{aligned} \Gamma \left( \overset{*}{\tilde{\rho}}, Id_E^* \right) \left( \overset{\circ}{\delta}_c \right) \left( (\rho, \eta) \overset{\circ}{\Gamma}_{bd} \right) &= \Gamma \left( \overset{*}{\tilde{\rho}}, Id_E^* \right) \left( \overset{*}{\delta}_c \right) \left( (\rho, \eta) \Gamma_{bd} \right) \\ &+ \frac{1}{4} \Gamma \left( \overset{*}{\tilde{\rho}}, Id_E^* \right) \left( \overset{*}{\delta}_c \right) \left( \left( \tilde{g}_{de} \circ h \circ \pi^* \right) \frac{\partial F_b}{\partial p_e} \right) \\ &- \frac{1}{4} \left( \tilde{g}_{ce} \circ h \circ \pi^* \right) \frac{\partial F_f}{\partial p_e} \frac{\partial}{\partial p_f} \left( (\rho, \eta) \Gamma_{bd} \right) \\ &- \frac{1}{16} \left( \tilde{g}_{ce} \circ h \circ \pi^* \right) \frac{\partial F_f}{\partial p_e} \frac{\partial}{\partial p_f} \left( \left( \tilde{g}_{de} \circ h \circ \pi^* \right) \frac{\partial F_b}{\partial p_e} \right), \end{aligned}$$

$$\begin{aligned}
\Gamma \left( \begin{smallmatrix} * \\ \tilde{\rho}, Id_E^* \end{smallmatrix} \right) \left( \begin{smallmatrix} * \\ \tilde{\delta}_d \end{smallmatrix} \right) ((\rho, \eta) \mathring{\Gamma}_{bc}) &= \Gamma \left( \begin{smallmatrix} * \\ \tilde{\rho}, Id_E^* \end{smallmatrix} \right) \left( \begin{smallmatrix} * \\ \tilde{\delta}_d \end{smallmatrix} \right) ((\rho, \eta) \Gamma_{bc}) \\
&+ \frac{1}{4} \Gamma \left( \begin{smallmatrix} * \\ \tilde{\rho}, Id_E^* \end{smallmatrix} \right) \left( \begin{smallmatrix} * \\ \tilde{\delta}_d \end{smallmatrix} \right) \left( \left( \tilde{g}_{ce} \circ h \circ \pi^* \right) \frac{\partial F_b}{\partial p_e} \right) \\
&- \frac{1}{4} \left( \tilde{g}_{de} \circ h \circ \pi^* \right) \frac{\partial F_f}{\partial p_e} \frac{\partial}{\partial p_f} ((\rho, \eta) \Gamma_{bc}) \\
&- \frac{1}{16} \left( \tilde{g}_{de} \circ h \circ \pi^* \right) \frac{\partial F_f}{\partial p_e} \frac{\partial}{\partial p_f} \left( \left( \tilde{g}_{ce} \circ h \circ \pi^* \right) \frac{\partial F_b}{\partial p_e} \right), \\
\left( L_{cd}^e \circ h \circ \pi^* \right) (\rho, \eta) \mathring{\Gamma}_{be} &= \left( L_{cd}^e \circ h \circ \pi^* \right) (\rho, \eta) \Gamma_{be} \\
&+ \left( L_{cd}^e \circ h \circ \pi^* \right) \left( \left( \tilde{g}_{fe} \circ h \circ \pi^* \right) \frac{\partial F_b}{\partial p_e} \right)
\end{aligned}$$

it results the conclusion of the theorem.

*q.e.d.*

**Proposition 8.1** *If  $\overset{*}{S}$  is the canonical  $(\rho, \eta)$ -semispray associated to the mechanical  $(\rho, \eta)$ -system  $\left( \left( \begin{smallmatrix} * \\ E, \pi, M \end{smallmatrix} \right), \overset{*}{F}_e, (\rho, \eta) \overset{*}{\Gamma} \right)$  and from locally invertible  $\mathbf{B}^{\mathbf{v}}$ -morphism  $(g, h)$ , then*

$$(8.8) \quad 2G_b = 2G_b \cdot M_b^b \circ h \circ \pi^* - \left( g^{ae} \circ h \circ \pi^* \right) p_e \left( \rho_a^i \circ h \circ \pi^* \right) \frac{\partial p_b}{\partial x^i}.$$

*Proof.* Since the Jacobian matrix of coordinates transformation is

$$\left\| \begin{array}{cc} M_a^{a'} \circ h \circ \pi^* & 0 \\ \left( \rho_a^i \circ h \circ \pi^* \right) \frac{\partial M_b^{a'} \circ \pi^*}{\partial x^i} p_a & M_b^b \circ \pi^* \end{array} \right\| = \left\| \begin{array}{cc} M_a^{a'} \circ h \circ \pi^* & 0 \\ \left( \rho_a^i \circ h \circ \pi^* \right) \frac{\partial p_b}{\partial x^i} & M_b^b \circ \pi^* \end{array} \right\|$$

and

$$\left\| \begin{array}{cc} M_a^{a'} \circ h \circ \pi^* & 0 \\ \left( \rho_a^i \circ h \circ \pi^* \right) \frac{\partial p_b}{\partial x^i} & M_b^b \circ \pi^* \end{array} \right\| \left( \begin{array}{c} \left( g^{ae} \circ h \circ \pi^* \right) p_e \\ -2 \left( G_b - \frac{1}{4} F_b \right) \end{array} \right) = \left( \begin{array}{c} \left( g^{a'e'} \circ h \circ \pi^* \right) p_{e'} \\ -2 \left( G_b - \frac{1}{4} F_b \right) \end{array} \right),$$

the conclusion results.

*q.e.d.*

In the following we consider a differentiable curve  $I \xrightarrow{\mathcal{C}} M$  and its  $(g, h)$ -lift  $\dot{c}$ .

**Definition 8.3** If it verifies the following equality:

$$(8.9) \quad \frac{d\dot{c}(t)}{dt} = \Gamma \left( \begin{smallmatrix} * \\ \tilde{\rho}, Id_E^* \end{smallmatrix} \right) \overset{*}{S}(\dot{c}(t)),$$

then we say that the curve  $\dot{c}$  is an integral curve of the  $(\rho, \eta)$ -semispray  $\overset{*}{S}$  of the dual mechanical  $(\rho, \eta)$ -system  $\left( \left( \begin{smallmatrix} * \\ E, \pi, M \end{smallmatrix} \right), \overset{*}{F}_e, (\rho, \eta) \overset{*}{\Gamma} \right)$ .

**Theorem 8.5** *The integral curves of the canonical  $(\rho, \eta)$ -semispray associated to the dual mechanical  $(\rho, \eta)$ -system  $\left( \left( \begin{smallmatrix} * \\ E, \pi, M \end{smallmatrix} \right), \overset{*}{F}_e, (\rho, \eta) \overset{*}{\Gamma} \right)$  and from locally invertible  $\mathbf{B}^{\mathbf{v}}$ -morphism  $(g, h)$ , are the  $(g, h)$ -lifts solutions of the equations:*

$$(8.10) \quad \frac{dp_b(t)}{dt} + 2G_b \circ \overset{*}{u}(c, \dot{c})(x(t)) = \frac{1}{2} F_b \circ \overset{*}{u}(c, \dot{c})(x(t)), \quad b \in \overline{1, r},$$

where  $x(t) = (\eta \circ h \circ c)(t)$ .

*Proof.* Since the equality

$$\frac{d\dot{c}(t)}{dt} = \Gamma \left( \tilde{\rho}, Id_E^* \right) S^* (\dot{c}(t))$$

is equivalent with

$$\begin{aligned} & \frac{d}{dt} ((\eta \circ h \circ c)^i(t), p_b(t)) = \\ & = (\rho_a^i \circ \eta \circ h \circ c(t) g^{ae} \circ h \circ c(t) p_e(t), -2 (G_b - \frac{1}{4} F_b) ((\eta \circ h \circ c)(t), p(t))), \end{aligned}$$

it results

$$\begin{aligned} \frac{dp_b(t)}{dt} + 2G_b(x(t), p(t)) &= \frac{1}{2} F_b(x(t), p(t)), \quad b \in \overline{1, r}, \\ \frac{dx^i(t)}{dt} &= \rho_a^i \circ \eta \circ h \circ c(t) g^{ae} \circ h \circ c(t) p_e(t), \end{aligned}$$

where  $x^i(t) = (\eta \circ h \circ c)^i(t)$ .

*q.e.d.*

**Definition 8.4** If  $S^*$  is a  $(\rho, \eta)$ -semispray, then the vector field

$$(8.11) \quad \left[ \begin{matrix} * \\ \mathbb{C}, S^* \end{matrix} \right]_{(\rho, \eta)TE^*} - S^*$$

will be called the *derivation of  $(\rho, \eta)$ -semispray  $S^*$* .

The  $(\rho, \eta)$ -semispray  $S^*$  will be called  $(\rho, \eta)$ -*spray* if there are verified the following conditions:

1.  $S^* \circ 0 \in C^1$ , where 0 is the null section;
2. Its derivation is the null vector field.

The  $(\rho, \eta)$ -semispray  $S^*$  will be called *quadratic  $(\rho, \eta)$ -spray* if there are verified the following conditions:

1.  $S^* \circ 0 \in C^2$ , where 0 is the null section;
2. Its derivation is the null vector field.

In particular, if  $(\rho, \eta) = (id_{TM}, Id_M)$  and  $(g, h) = (Id_E, Id_M)$ , then we obtain the *spray* and the *quadratic spray* which is similar with the classical spray and quadratic spray.

**Theorem 8.6** *If  $S$  is the canonical  $(\rho, \eta)$ -spray associated to the dual mechanical  $(\rho, \eta)$ -system  $\left( \left( \begin{matrix} * \\ E, \pi, M \end{matrix} \right), F_e^*, (\rho, \eta) \Gamma^* \right)$  and from locally invertible  $\mathbf{B}^v$ -morphism  $(g, h)$ , then*

$$\begin{aligned} 2(G_b - \frac{1}{4} F_b) &= (\rho, \eta) \Gamma_{bc} \left( g^{cf} \circ h \circ \pi^* \right) p_f \\ &+ \frac{1}{2} \left( g^{de} \circ h \circ \pi^* \right) p_e \left( L_{dc}^a \circ h \circ \pi^* \right) \left( \tilde{g}_{ab} \circ h \circ \pi^* \right) \left( g^{cf} \circ h \circ \pi^* \right) p_f \\ &- \frac{1}{2} \left( \rho_c^j \circ h \circ \pi^* \right) \frac{\partial (g^{ae} \circ h \circ \pi^*)}{\partial x^j} p_e \left( \tilde{g}_{ab} \circ h \circ \pi^* \right) \left( g^{cf} \circ h \circ \pi^* \right) p_f \\ &+ \frac{1}{2} \left( g^{ae} \circ h \circ \pi^* \right) p_e \left( \rho_a^i \circ h \circ \pi^* \right) \frac{\partial (\tilde{g}_{bc} \circ h \circ \pi^*)}{\partial x^i} \left( g^{cf} \circ h \circ \pi^* \right) p_f \end{aligned} \quad (8.12)$$

We obtain the spray

$$\begin{aligned}
(8.13) \quad S &= (g^{ae} \circ h \circ \pi) p_e \tilde{\partial}_a^* - (\rho, \eta) \Gamma_{bc} (g^{cf} \circ h \circ \pi) p_f \tilde{\partial}^{\cdot b} \\
&\quad - \frac{1}{2} (g^{de} \circ h \circ \pi) p_e (L_{dc}^a \circ h \circ \pi) (\tilde{g}_{ab} \circ h \circ \pi) (g^{cf} \circ h \circ \pi) p_f \tilde{\partial}^{\cdot b} \\
&\quad + \frac{1}{2} (\rho_c^j \circ h \circ \pi) \frac{\partial(g^{ae} \circ h \circ \pi)}{\partial x^j} p_e (\tilde{g}_{ab} \circ h \circ \pi) (g^{cf} \circ h \circ \pi) p_f \tilde{\partial}^{\cdot b} \\
&\quad - \frac{1}{2} (g^{ae} \circ h \circ \pi) p_e (\rho_a^i \circ h \circ \pi) \frac{\partial(\tilde{g}_{bc} \circ h \circ \pi)}{\partial x^i} (g^{cf} \circ h \circ \pi) p_f \tilde{\partial}^{\cdot b}
\end{aligned}$$

This spray will be called the canonical  $(\rho, \eta)$ -spray associated to the dual mechanical system  $\left( \left( \overset{*}{E}, \overset{*}{\pi}, M \right), \overset{*}{F}_e, (\rho, \eta) \overset{*}{\Gamma} \right)$  and from locally invertible  $\mathbf{B}^{\mathbf{v}}$ -morphism  $(g, h)$ .

In particular, if  $(\rho, \eta) = (id_{TM}, Id_M)$  and  $(g, h) = (Id_E, Id_M)$ , then we get the canonical spray associated to connection  $\Gamma$  which is similar with the classical canonical spray associated to connection  $\Gamma$ .

*Proof.* Since

$$\begin{aligned}
\left[ \overset{*}{\mathbb{C}}, \overset{*}{S} \right]_{(\rho, \eta)TE}^* &= \left[ p_a \tilde{\partial}^{\cdot a}, (g^{be} \circ h \circ \pi) p_e \tilde{\partial}_b^* \right]_{(\rho, \eta)TE}^* - 2 \left[ p_a \tilde{\partial}^{\cdot a}, (G_b - \frac{1}{4} F_b) \tilde{\partial}^{\cdot b} \right]_{(\rho, \eta)TE}^*, \\
\left[ p_a \tilde{\partial}^{\cdot a}, (g^{be} \circ h \circ \pi) p_e \tilde{\partial}_b^* \right]_{(\rho, \eta)TE}^* &= (g^{be} \circ h \circ \pi) p_e \tilde{\partial}_b^*
\end{aligned}$$

and

$$\left[ p_a \tilde{\partial}^{\cdot a}, (G_b - \frac{1}{4} F_b) \tilde{\partial}^{\cdot b} \right]_{(\rho, \eta)TE}^* = p_a \frac{\partial (G_b - \frac{1}{4} F_b)}{\partial p_a} \tilde{\partial}^{\cdot b} - \left( G_b - \frac{1}{4} F_b \right) \tilde{\partial}^{\cdot b}$$

it results that

$$(S_1) \quad \left[ \overset{*}{\mathbb{C}}, \overset{*}{S} \right]_{(\rho, \eta)TE}^* - \overset{*}{S} = 2 \left( -p_f \frac{\partial (G_b - \frac{1}{4} F_b)}{\partial p_f} + 2 \left( G_b - \frac{1}{4} F_b \right) \right) \tilde{\partial}^{\cdot b}$$

Using equality (8.3), it results that

$$\begin{aligned}
(S_2) \quad \frac{\partial (G_b - \frac{1}{4} F_b)}{\partial p_f} &= (\rho, \eta) \Gamma_{bc} (g^{cf} \circ h \circ \pi) \\
&\quad + \frac{1}{2} (g^{de} \circ h \circ \pi) p_e (L_{dc}^a \circ h \circ \pi) (\tilde{g}_{ab} \circ h \circ \pi) (g^{cf} \circ h \circ \pi) \\
&\quad - \frac{1}{2} (\rho_c^j \circ h \circ \pi) \frac{\partial(g^{ae} \circ h \circ \pi)}{\partial x^j} p_e (\tilde{g}_{ab} \circ h \circ \pi) (g^{cf} \circ h \circ \pi) \\
&\quad + \frac{1}{2} (g^{ae} \circ h \circ \pi) p_e (\rho_a^i \circ h \circ \pi) \frac{\partial(\tilde{g}_{bc} \circ h \circ \pi)}{\partial x^i} (g^{cf} \circ h \circ \pi)
\end{aligned}$$

Using equalities  $(S_1)$  and  $(S_2)$ , it results the conclusion of the theorem. *q.e.d.*

**Theorem 8.7** All  $(g, h)$ -lifts solutions of the following system of equations:

$$\begin{aligned}
(8.14) \quad & \frac{dp_b}{dt} + (\rho, \eta) \Gamma_{bc} \left( g^{cf} \circ h \circ \pi^* \right) p_f \\
& + \frac{1}{2} \left( g^{de} \circ h \circ \pi^* \right) p_e \left( L_{dc}^b \circ h \circ \pi^* \right) \left( \tilde{g}_{ba} \circ h \circ \pi^* \right) \left( g^{cf} \circ h \circ \pi^* \right) p_f \\
& - \frac{1}{2} \left( \rho_c^j \circ h \circ \pi^* \right) \frac{\partial \left( g^{be} \circ h \circ \pi^* \right)}{\partial x^j} p_e \left( \tilde{g}_{ba} \circ h \circ \pi^* \right) \left( g^{cf} \circ h \circ \pi^* \right) p_f \\
& + \frac{1}{2} \left( g^{ae} \circ h \circ \pi^* \right) p_e \left( \rho_a^i \circ h \circ \pi^* \right) \frac{\partial \left( \tilde{g}_{bc} \circ h \circ \pi^* \right)}{\partial x^i} \left( g^{cf} \circ h \circ \pi^* \right) p_f = 0,
\end{aligned}$$

are the integral curves of canonical  $(\rho, \eta)$ -spray associated to the dual mechanical  $(\rho, \eta)$ -system  $\left( \left( \overset{*}{E}, \overset{*}{\pi}, M \right), \overset{*}{F}_e, (\rho, \eta) \overset{*}{\Gamma} \right)$  and from locally invertible  $\mathbf{B}^\vee$ -morphism  $(g, h)$ .

## 9 A Hamiltonian formalism for Hamilton mechanical $(\rho, \eta)$ -systems

Let  $\left( \left( \overset{*}{E}, \overset{*}{\pi}, M \right), \overset{*}{F}_e, H \right)$  be an arbitrarily Hamilton mechanical  $(\rho, \eta)$ -system.

Let  $(d\tilde{z}^a, d\tilde{p}_a)$  be the natural dual  $(\rho, \eta)$ -base of the natural  $(\rho, \eta)$ -base  $\left( \overset{*}{\tilde{\partial}}_a, \overset{\cdot}{\tilde{\partial}}^a \right)$ .

It is very important to remark that the 1-forms  $d\tilde{z}^a, d\tilde{p}_a$ ,  $a \in \overline{1, p}$  are not the differentials of coordinates functions as in the classical case, but we will use the same notations. In this case

$$(d\tilde{z}^a) \neq d^{(\rho, \eta)TE^*}(\tilde{z}^a),$$

where  $d^{(\rho, \eta)TE^*}$  is the exterior differentiation operator associated to exterior differential  $\mathcal{F} \left( \overset{*}{E} \right)$ -algebra

$$\left( \Lambda \left( (\rho, \eta) TE^*, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right), +, \cdot, \wedge \right).$$

Let  $H$  be a regular Hamiltonian and let  $(g, h)$  be a locally invertible  $\mathbf{B}^\vee$ -morphism of  $\left( \overset{*}{E}, \overset{*}{\pi}, M \right)$  source and  $(E, \pi, M)$  target.

**Definition 9.1** The 1-form

$$(9.1) \quad \theta_H = \left( \tilde{g}_{ae} \circ h \circ \pi^* \right) H^e d\tilde{z}^a$$

will be called the 1-form of Poincaré-Cartan type associated to the regular Hamiltonian  $H$  and from locally invertible  $\mathbf{B}^\vee$ -morphism  $(g, h)$ .

We obtain easily:

$$(9.2) \quad \theta_H \left( \overset{*}{\tilde{\partial}}_b \right) = \left( \tilde{g}_{be} \circ h \circ \pi^* \right) \cdot H^e, \quad \theta_H \left( \overset{\cdot}{\tilde{\partial}}^b \right) = 0.$$



**Definition 9.2** The 2-form

$$\omega_H = d^{(\rho, \eta)TE^*} \theta_H$$

will be called the 2-form of Poincaré-Cartan type associated to the Hamiltonian  $H$  and to the locally invertible  $\mathbf{B}^v$ -morphism  $(g, h)$ .

By the definition of  $d^{(\rho, \eta)TE^*}$ , we obtain:

$$(9.3) \quad \begin{aligned} \omega_H(U, V) &= \Gamma \left( \tilde{\rho}, Id_E^* \right) (U) (\theta_H(V)) \\ &\quad - \Gamma \left( \tilde{\rho}, Id_E^* \right) (V) (\theta_H(U)) - \theta_H \left( [U, V]_{(\rho, \eta)TE^*} \right), \end{aligned}$$

for any  $U, V \in \Gamma \left( (\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E \right)$ .

**Definition 9.3** The real function

$$(9.4) \quad \mathcal{E}_H = p_a H^a - H$$

will be called the energy of regular Hamiltonian  $H$ .

**Theorem 9.1** The equation

$$(9.5) \quad i_S(\omega_H) = -d^{(\rho, \eta)TE^*}(\mathcal{E}_H), \quad S \in \Gamma \left( (\rho, \eta)TE^*, (\rho, \eta)\tau_E^*, E \right),$$

has an unique solution  $\tilde{S}_H^*(g, h)$  of the type:

$$(9.6) \quad \left( g^{ae} \circ h \circ \pi^* \right) p_e \tilde{\partial}_a^* - 2 \left( G_a - \frac{1}{4} F_a \right) \tilde{\partial}^a,$$

where

$$(9.7) \quad -2 \left( G_a - \frac{1}{4} F_a \right) = E_b(H, g, h) \tilde{H}_{ae} \left( g^{eb} \circ h \circ \pi^* \right)$$

and

$$(9.8) \quad \begin{aligned} E_b(H, g, h) &= \left( \rho_b^i \circ h \circ \pi^* \right) H_i - \left( \rho_b^i \circ h \circ \pi^* \right) p_a H_i^a \\ &\quad - \left( g^{df} \circ h \circ \pi^* \right) p_f \left( \rho_d^i \circ h \circ \pi^* \right) \frac{\partial \left( \left( \tilde{g}_{be} \circ h \circ \pi^* \right) H^e \right)}{\partial x^i} \\ &\quad + \left( g^{df} \circ h \circ \pi^* \right) p_f \left( \rho_b^i \circ h \circ \pi^* \right) \frac{\partial \left( \left( \tilde{g}_{de} \circ h \circ \pi^* \right) H^e \right)}{\partial x^i} \\ &\quad + \left( g^{df} \circ h \circ \pi^* \right) p_f \left( L_{db}^c \circ h \circ \pi^* \right) \left( \tilde{g}_{ce} \circ h \circ \pi^* \right) H^e \end{aligned}$$

$\tilde{S}_H^*(g, h)$  will be called the canonical  $(\rho, \eta)$ -semispray associated to the Hamilton mechanical  $(\rho, \eta)$ -system  $\left( \left( E, \pi^*, M \right), F_e^*, H \right)$  and from locally invertible  $\mathbf{B}^v$ -morphism  $(g, h)$ .

*Proof.* We obtain that

$$i_{\tilde{S}_H^*}(\omega_H) = -d^{(\rho, \eta)TE^*}(\mathcal{E}_H)$$

if and only if

$$\omega_H \left( \overset{*}{S}, X \right) = -\Gamma \left( \overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}} \right) (X) (\mathcal{E}_H),$$

for any  $X \in \Gamma \left( (\rho, \eta) T\overset{*}{E}, (\rho, \eta) \tau_{\overset{*}{E}}, \overset{*}{E} \right)$ .

Particularly, we obtain:

$$\omega_L \left( \overset{*}{S}, \overset{*}{\tilde{\partial}_b} \right) = -\Gamma \left( \overset{*}{\tilde{\rho}}, Id_{\overset{*}{E}} \right) \left( \overset{*}{\tilde{\partial}_b} \right) (\mathcal{E}_H).$$

If we expand this equality, we obtain

$$\begin{aligned} & \left( g^{df} \circ h \circ \overset{*}{\pi} \right) p_f \left[ \left( \rho_d^i \circ h \circ \overset{*}{\pi} \right) \frac{\partial \left( (\tilde{g}_{be} \circ h \circ \overset{*}{\pi}) H^e \right)}{\partial x^i} - \left( \rho_b^i \circ h \circ \overset{*}{\pi} \right) \frac{\partial \left( (\tilde{g}_{de} \circ h \circ \overset{*}{\pi}) H^e \right)}{\partial x^i} \right. \\ & \left. - \left( L_{db}^c \circ h \circ \overset{*}{\pi} \right) \left( \tilde{g}_{ce} \circ h \circ \overset{*}{\pi} \right) H^e \right] - 2 \left( G_b - \frac{1}{4} F_b \right) \left( \tilde{g}_{ae} \circ h \circ \overset{*}{\pi} \right) \cdot H^{eb} \\ & = \left( \rho_b^i \circ h \circ \overset{*}{\pi} \right) L_i - \left( \rho_b^i \circ h \circ \overset{*}{\pi} \right) \frac{\partial (p_a H^a)}{\partial x^i}. \end{aligned}$$

After some calculations, we obtain the conclusion of the theorem.

*q.e.d.*

**Theorem 9.2** If  $\overset{*}{S}_H(g, h)$  is the canonical  $(\rho, \eta)$ -semispray associated to the Hamilton mechanical  $(\rho, \eta)$ -system  $\left( \left( \overset{*}{E}, \overset{*}{\pi}, M \right), \overset{*}{F}_e, H \right)$  and from locally invertible  $\mathbf{B}^v$ -morphism  $(g, h)$ , then the real local functions

$$\begin{aligned} (\rho, \eta) \Gamma_{bc} &= -\frac{1}{2} \left( \tilde{g}_{cd} \circ h \circ \overset{*}{\pi} \right) \frac{\partial \left( E_b(H, g, h) \tilde{H}_{ae} (g^{eb} \circ h \circ \overset{*}{\pi}) \right)}{\partial p_d} \\ &\quad - \frac{1}{2} \left( g^{de} \circ h \circ \overset{*}{\pi} \right) p_e \left( L_{dc}^f \circ h \circ \overset{*}{\pi} \right) \left( \tilde{g}_{fb} \circ h \circ \overset{*}{\pi} \right) \\ &\quad + \frac{1}{2} \left( \rho_c^j \circ h \circ \overset{*}{\pi} \right) \frac{\partial \left( g^{be} \circ h \circ \overset{*}{\pi} \right)}{\partial x^j} p_e \left( \tilde{g}_{ba} \circ h \circ \overset{*}{\pi} \right) \\ &\quad - \frac{1}{2} \left( g^{de} \circ h \circ \overset{*}{\pi} \right) p_e \left( \rho_d^i \circ h \circ \overset{*}{\pi} \right) \frac{\partial \left( \tilde{g}_{bc} \circ h \circ \overset{*}{\pi} \right)}{\partial x^i} \end{aligned} \quad (9.9)$$

are the components of a  $(\rho, \eta)$ -connection  $(\rho, \eta) \Gamma$  for the vector bundle  $\left( \overset{*}{E}, \overset{*}{\pi}, M \right)$  which will be called the  $(\rho, \eta)$ -connection associated to the Hamilton mechanical  $(\rho, \eta)$ -system  $\left( \left( \overset{*}{E}, \overset{*}{\pi}, M \right), \overset{*}{F}_e, H \right)$  and from locally invertible  $\mathbf{B}^v$ -morphism  $(g, h)$ .

**Theorem 9.3** The parallel  $(g, h)$ -lifts with respect to  $(\rho, \eta)$ -connection  $(\rho, \eta) \Gamma$  are the integral curves of the canonical  $(\rho, \eta)$ -semispray associated to the Hamilton mechanical  $(\rho, \eta)$ -system  $\left( \left( \overset{*}{E}, \overset{*}{\pi}, M \right), \overset{*}{F}_e, H \right)$  and from locally invertible  $\mathbf{B}^v$ -morphism  $(g, h)$ .

**Definition 9.4** The equations

$$(9.10) \quad \frac{dp_a(t)}{dt} - E_b(H, g, h) \tilde{H}_{ae} \left( g^{eb} \circ h \circ \overset{*}{\pi} \right) \circ u(c, \dot{c})(x(t)) = 0,$$

where  $x(t) = \eta \circ h \circ c(t)$ , will be called the *equations of Hamilton-Jacobi type associated to the Hamilton mechanical  $(\rho, \eta)$ -system  $\left( \left( \overset{*}{E}, \overset{*}{\pi}, M \right), \overset{*}{F}_e, H \right)$  and from locally invertible  $\mathbf{B}^v$ -morphism  $(g, h)$* .

*Remark 9.1* The integral curves of the canonical  $(\rho, \eta)$ -semispray associated to the Hamilton mechanical  $(\rho, \eta)$ -system  $\left(\left(\begin{smallmatrix} * \\ E, \pi, M \end{smallmatrix}\right), \begin{smallmatrix} * \\ F_e, H \end{smallmatrix}\right)$  and from locally invertible  $\mathbf{B}^v$ -morphism  $(g, h)$  are the  $(g, h)$ -lifts solutions for the equations of Hamilton-Jacobi type (9.10).

Using our theory, we obtain the following

**Theorem 9.4** *If  $K$  is a Cartan fundamental function, then the geodesics on the manifold  $M$  are the curves such that the components of their  $(g, h)$ -lifts are solutions for the equations of Hamilton-Jacobi type (9.10).*

Therefore, it is natural to propose to extend the study of Cartan geometry from the dual of the Lie algebroid  $((TM, \tau_M, M), [\cdot, \cdot], (Id_{TM}, Id_M))$ , to the dual of an arbitrary (generalized) Lie algebroid  $((E, \pi, M), [\cdot, \cdot]_{E, h}, (\rho, \eta))$ .

## References

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